6 Self-consistent mean field electrodynamics in two and three dimensions

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The status of the self-consistent theory of mean field electrodynamics for incompressible MHD is reviewed. This discussion treats the calculation of the transport of magnetic potential or flux in two and three dimensions, the α-effect in three dimensions, and the transport of momentum in two dimensions. Physical interpretations and connections to numerical experiments are emphasized.

1. Introduction

Mean field electrodynamics is concerned with the application of the techniques of mean field or quasi-linear theory to the derivation of local, turbulent transport equations for macroscopic quantities (such as the average magnetic field $\langle \mathbf{B} \rangle$) in MHD and other plasma models. The most well-known products of mean field electrodynamics are the α and

β coefficients for the evolution of $\langle \mathbf{B} \rangle$ in 3D, i.e.,

$$\frac{\partial}{\partial t}\langle \mathbf{B} \rangle = \boldsymbol{\nabla} \times \alpha \langle \mathbf{B} \rangle + (\eta + \beta)\nabla^2 \langle \mathbf{B} \rangle, \tag{1}$$

and the turbulent resistivity η_T in 2D, i.e.,

$$\frac{\partial}{\partial t}\langle A \rangle(x,t) = \eta_T \frac{\partial^2}{\partial x^2}\langle A \rangle, \tag{2}$$

where A is the magnetic potential. (In (1) and (2) we have considered the simplest case, in which the underlying turbulence is assumed to be homogeneous and isotropic, and for which α, β and η_T are constant.) Of course, α is the familiar "dynamo coefficient", which aims to capture, in a local transport coefficient, the fundamental process of amplification of field by cyclonic turbulence. Here α is a pseudo-scalar and nearly always depends on the turbulence helicity; β and η_T typically depend on the turbulence energy, but may involve other quantities. In addition, α, β and η_T each involve a field-fluid correlation time.

The practice of mean field electrodynamics is a well-developed art form, the fundamentals of which are set forth in the classic monograph by Moffatt (1978). Until recently, mean field electrodynamics calculations were kinematic in character, and the fluid energy spectrum and, more subtly, the field-fluid correlation time, were taken as given. The effects of the small-scale magnetic field on either the transport coefficients or on the correlation time were almost always neglected. An important exception to this trend was the work of Pouquet *et al.* (1976, 1978). However, in a seminal paper, Cattaneo and Vainshtein (1991) convincingly suggested that for high R_m (the magnetic Reynolds number), small-scale magnetic field effects on mean field evolution are *not* negligible even for very weak values of the large-scale field, and that, consequently, diffusion of the mean flux is strongly reduced, or "quenched", in comparison to its kinematic value. The quench reflects the presence of a "dynamical memory" on the part of the magnetic flux with respect to its location relative to the fluid. This type of analysis was rapidly extended to the α-effect in 3D incompressible MHD and various other systems. As the results of these investigations shook the foundations of the prevailing concepts of magnetic dynamo and field diffusion processes at high R_m, they quite naturally also engendered significant controversy.

It is the purpose of this paper to present a review of the theory of self-consistent mean field electrodynamics at high R_m — the regime of interest for astrophysical plasmas. There is no pretense of completeness, the "last word", or "neutrality" in this review. Rather, we seek to set forth our own understanding of the current status of the field, the principal ideas and results, and the main unresolved physics issues. We strive throughout to elucidate the basic physics and to make connections with relevant computational studies. In this paper, we discuss the application of mean field electrodynamics to the turbulent diffusion of magnetic fields in two and three dimensions, to the α-effect (necessarily in 3D) and to momentum transport and the effect of flow shear in 2D. In all cases, we assume a mean magnetic field (in most cases weak), which is distorted and stretched by turbulent motions. This distortion is the origin of the small-scale magnetic field, which grows rapidly in comparison to the timescale upon which the mean field evolves. The evolution of the mean field (either via diffusion or cyclonic distortion) is then considered in the presence of *both* small-scale fluid velocity and magnetic perturbations. Hence, we refer to these calculations as examples of *self-consistent* mean field electrodynamics.

The remainder of this review is organized as follows. Section 2 presents the theory of magnetic flux and field diffusion (i.e., the calculation of η_T and β) in 2D and 3D. We demonstrate that treating the small-scale magnetic turbulence on a footing equal to that of the velocity fluctuations, along with the constraint of mean square magnetic potential conservation, together imply that magnetic diffusion is quenched in comparison to kinematic predictions. We demonstrate that this quench is critically dependent on the magnetic Reynolds number. New results on the effects of boundaries and scale-to-scale coupling of magnetic potential are presented. Magnetic diffusion in reduced and full 3D MHD is also discussed. Section 3 discusses α-quenching in 3D MHD. The basic theory of the α-effect is reviewed. The incidence of the quench is related to the combined effects of the "back-α" effect of small-scale magnetic fields and the conservation of magnetic helicity. The implications of related computational studies are discussed and assessed. We also address some of the contradictory claims and controversy surrounding this topic. Section 4 presents the mean field theory of momentum transport in 2D and the effect of mean shear flow on flux diffusion in 2D. Scalings of the effective turbulent viscosity and resistivity are derived for the strong shear and strong field limits. The crucial role of "Alfvénization" is identified. The relevance of these calculations to the mechanism of the interface dynamo at the boundary of the solar convection zone and tachocline is discussed. Although effects such as rotation and compressibility are obviously of importance in astrophysical contexts, many of the fundamental issues of turbulent transport can be considered within the framework of incompressible MHD, and hence we limit our discussion to this case throughout.

2. Turbulent diffusion of magnetic fields

2.1. Overview

In this section, we review the status of the mean field theory of diffusion of magnetic fields in 2D and 3D incompressible MHD. Attention is focused primarily on the simpler 2D problem, for which the effects of diffusion are not entangled with those of field growth through dynamo action. Also, and perhaps surprisingly, there are many close analogies between mean flux diffusion in 2D and that of the mean field α-effect in 3D. We discuss the similarities and differences between the two problems, with the goal of developing insight into the more interesting (but difficult!) 3D α-effect problem from the simpler 2D diffusion problem. Throughout our discussion, magnetic Prandtl number (P_m) of unity (i.e., $\nu = \eta$) and periodic boundary conditions are assumed, unless otherwise explicitly noted, although it is worth mentioning that interesting questions arise as to the resulting behavior when either of these assumptions are relaxed.

2.2. Flux diffusion in 2D—basic model and concepts

The familiar equations of 2D MHD are

$$\frac{\partial A}{\partial t} + (\nabla\psi \times \hat{\mathbf{z}}) \cdot \nabla A = \eta \nabla^2 A, \tag{3}$$

$$\frac{\partial}{\partial t}\nabla^2\psi + (\nabla\psi \times \hat{\mathbf{z}}) \cdot \nabla\nabla^2\psi = (\nabla A \times \hat{\mathbf{z}}) \cdot \nabla\nabla^2 A + \nu\nabla^2\nabla^2\psi, \tag{4}$$

where A is the magnetic potential ($\mathbf{B} = \nabla \times A\hat{\mathbf{z}}$), ψ is the velocity stream function ($\mathbf{v} = \nabla \times \psi\hat{\mathbf{z}}$), η is the resistivity, ν is the viscosity and $\hat{\mathbf{z}}$ is the unit vector orthogonal to the plane of motion. We shall consider the case where the mean magnetic field is in the y-direction, and is a slowly varying function of x. Equations (3) and (4) have non-dissipative quadratic invariants, the energy $E = \int[(\nabla A)^2 + (\nabla\psi)^2]\mathrm{d}^2x$, mean-square magnetic potential $H_A = \int A^2\,\mathrm{d}^2x$ and cross helicity $H_c = \int \nabla A \cdot \nabla\psi\,\mathrm{d}^2x$. Throughout this paper, we take $H_c = 0$ *ab initio*, so there is no net Alfvénic alignment in the MHD turbulence considered here. The effects of cross helicity on MHD turbulence are discussed by Grappin *et al*. (1983).

The basic dynamics of 2D MHD turbulence are well understood. For large-scale stirring, energy is self-similarly transferred to small scales and eventual dissipation via an Alfvénized cascade, as originally suggested by Kraichnan (1965) and Iroshnikov (1964), and clearly

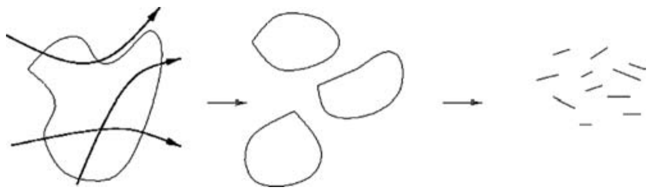

Figure 1. Forward transfer: fluid eddies chop up scalar A.

demonstrated in simulations by Biskamp and Welter (1989). The Kraichnan-Iroshnikov spectrum for the MHD turbulence cascade is the same in 2D as in 3D. This cascade may manifest anisotropy in the presence of a strong mean field in 3D, as predicted by Goldreich and Sridhar (1995, 1997). Mean square magnetic potential H_A, on the other hand, tends to accumulate at (or cascade toward) large scales, as is easily demonstrated by equilibrium statistical mechanics for non-dissipative 2D MHD (Fyfe and Montgomery, 1976). Here, H_c is the second conserved quadratic quantity (in addition to energy), which thus suggests a dual cascade. In 2D, the mean field quantity of interest is the spatial flux of magnetic potential $\Gamma_A = \langle v_x A \rangle$. An essential element of the physics of Γ_A is the competition between advection of scalar potential by the fluid, and the tendency of the flux A to coalesce at large scales. The former is, in the absence of back-reaction, simply a manifestation of the fact that turbulence tends to strain, mix and otherwise "chop up" a passive scalar field, thus generating small-scale structure (see Fig. 1). The latter manifests the fact that A is *not* a passive scalar, and that it resists mixing by the tendency to coagulate on large scales (see Fig. 2) (Riyopoulos *et al.*, 1982). The inverse cascade of A^2, like the phenomenon of magnetic island coalescence, is ultimately rooted in the fact that like-signed current filaments attract. Not surprisingly then, the velocity field drives a positive potential diffusivity (turbulent resistivity), while the magnetic field perturbations drive a *negative* potential diffusivity. Thus, we may anticipate a relation for the turbulent resistivity of the form $\eta_T \sim \langle v^2 \rangle - \langle B^2 \rangle$, a considerable departure from expectations based upon kinematic models. A similar competition between mixing and

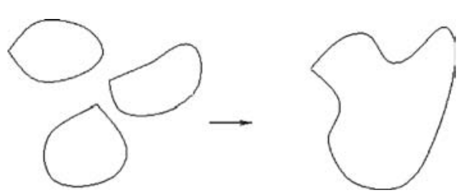

Figure 2. Inverse transfer: current filaments and A-blobs attract and coagulate.

coalescence appears in the spectral dynamics. Note also that η_T vanishes for turbulence at Alfvénic equipartition (i.e., $\langle v^2 \rangle = \langle B^2 \rangle$). Since the presence of even a weak mean magnetic field will naturally convert some of the fluid eddies to Alfvén waves, it is thus not entirely surprising that questions arise as to the possible reduction or "quenching" of the magnetic diffusivity relative to expectations based upon kinematics. Also, note that any such quenching is intrinsically a synergistic consequence of both:

(i) the competition between flux advection and flux coalescence intrinsic to 2D MHD;
(ii) the tendency of a mean magnetic field to "Alfvénize" the turbulence.

The close correspondence between the problems of 2D flux diffusion and that of the 3D mean field electromotive force, summarized in Table 1, is remarkable. Both seek a representation of a mean product of fluid and magnetic fluctuations (i.e., the mean e.m.f. $\boldsymbol{\mathcal{E}}$ in 3D, Γ_A in 2D) in terms of local transport coefficients, namely α and β in 3D and η_T in 2D. In each case, the magnetic dynamics are critically constrained by the conservation, up to resistive dissipation, of magnetic helicity in 3D and of H_A in 2D. Both magnetic helicity and H_A inverse cascade to large scales, and thus produce an interesting dual cascade, since energy flows to small scales in each case. The inverse cascade of magnetic helicity and mean-square potential underpin the appearance of magnetic "back-reaction" contributions to α and η_T. In particular, $\alpha \sim (\langle \mathbf{v} \cdot \boldsymbol{\omega} \rangle - \langle \mathbf{B} \cdot \mathbf{J} \rangle)$, while $\eta_T \sim \langle v^2 \rangle - \langle B^2 \rangle$. Thus, both tend to vanish for fully Alfvénized

Table 1. Table of analogies between calculations of 3D mean e.m.f. and 2D mean potential transport

3D Mean EMF	2D Mean Potential Flux
$\boldsymbol{\mathcal{E}} = \langle \mathbf{v} \times \mathbf{B} \rangle$	$\Gamma_A = \langle v_x A \rangle$
$\boldsymbol{\mathcal{E}} = \alpha \langle \mathbf{B} \rangle - \beta \langle \mathbf{J} \rangle$	$\Gamma_A = -\eta_T \, \partial \langle A \rangle / \partial x$
invariant $\rightarrow$ Helicity	invariant $\rightarrow$ mean-square magnetic potential
$\int \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3 x$	$\int A^2 \, \mathrm{d}^2 x$
inverse cascade of magnetic helicity	inverse cascade of H_A
back $\alpha \sim \langle \mathbf{B} \cdot \mathbf{J} \rangle$	negative diffusivity $\sim \langle B^2 \rangle$
$\langle \mathbf{B} \cdot \mathbf{J} \rangle$ from helicity balance	$\langle A^2 \rangle$ from H_A balance
α quenching	η_T quenching
β-quenching	

turbulence. This trend, then, naturally suggests the possibility of both α-quenching in 3D, and magnetic diffusivity quenching in 2D. Of course, there are crucial *differences* between the two problems. Obviously, in 2D only decay of the magnetic field is possible, whereas 3D admits the possibility of dynamo growth. Furthermore, magnetic helicity and α (the pertinent quantities in 3D) are pseudo-scalars while H_A and η_T are scalars; thus, the effect of helicity conservation on β, the magnetic diffusivity in three dimensions, remains far from clear.

An important element of the basic physics, common to both problems, is the process of "Alfvénization", whereby eddy energy is converted to Alfvén wave energy. This may be thought of as a physical perspective on the natural trend of MHD turbulence toward an approximate balance between fluid and magnetic energies, for $P_m \sim 1$. Note also that Alfvénization may be thought of as the development of a *dynamical memory*, which constrains and limits the cross-phase between v_x and A. This is readily apparent from the fact that $\langle v_x A \rangle$ vanishes for Alfvén waves in the absence of resistive dissipation. For Alfvén waves then, flux diffusion is directly proportional to resistive dissipation, an unsurprising conclusion for cross-field transport of flux which is, in turn, frozen into the fluid, up to η. As we shall soon see, the final outcome of the quenching calculation also reveals an explicit proportionality of η_T to η. For small η, then, Γ_A will be quenched. Another perspective on Alfvénization comes from the studies of Lyapunov exponents of fluid elements in MHD turbulence (Cattaneo *et al.*, 1996). These showed that as small-scale magnetic fields are amplified and react back on the flow, Lyapunov exponents drop precipitously, so that chaos is suppressed. This observation is consistent with the notion of the development of a dynamical memory, discussed above.

2.3. Mean field electrodynamics for $\langle A \rangle$ in 2D

In this section, we discuss the mean field theory of flux diffusion in 2D. In the discussion of the calculation of Γ_A, we do not address the relationship between the turbulent velocity field and the mechanisms by which the turbulence is excited or stirred. However, a weak large-scale field (the transport of which is the process to be studied) will be violently stretched and distorted, resulting in the rapid generation of a spectrum of magnetic turbulence. As discussed above, magnetic turbulence will likely tend to retard and impede the diffusion of large-scale magnetic fields. This, of course, is the crux of the matter, as Γ_A depends on the full spectrum arising from the external excitation and the back-reaction of the magnetic field, so the net imbalance of $\langle v^2 \rangle$ and $\langle B^2 \rangle$ determines the degree of η_T quenching. Leverage on $\langle B^2 \rangle$ is obtained by considering the evolution of mean-square magnetic potential density $\mathcal{H}_A$. In particular, the conservation of $H_A = \int \mathcal{H}_A \mathrm{d}^2x$ straightforwardly yields the identity

$$\frac{1}{2}\frac{\partial H_A}{\partial t} = -\Gamma_A \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle, \tag{5}$$

where the surface terms vanish for periodic boundaries. For stationary turbulence, then, this gives

$$\langle B^2 \rangle = -\frac{\Gamma_A}{\eta}\frac{\partial \langle A \rangle}{\partial x} = \frac{\eta_T}{\eta}\left(\frac{\partial \langle A \rangle}{\partial x}\right)^2, \tag{6}$$

which is the well-known Zeldovich (1957) theorem ($\langle B^2 \rangle / \langle B \rangle^2 = \eta_T / \eta$) for 2D MHD. The physics of the Zeldovich theorem is discussed further in the Appendix. The key message here is that when a weak mean magnetic field is coupled to a turbulent 2D flow, a *large mean-square fluctuation level can result*, on account of stretching iso-A or flux contours by the flow. However, while the behavior of $\langle B^2 \rangle$ is clear, we shall see that it is really $\langle B^2 \rangle_{\mathbf{k}}$ that enters the calculation of Γ_A, via a spectral sum.

To calculate Γ_A, standard closure methods (see, for example, Pouquet, 1978 or McComb, 1990) yield

$$\Gamma_A = \sum_{\mathbf{k}'} [v_x(-\mathbf{k}')\delta A(\mathbf{k}') - B_x(-\mathbf{k}')\delta\psi(\mathbf{k}')] = \sum_{\mathbf{k}'} \Gamma_A(\mathbf{k}'), \tag{7}$$

where $\delta A(\mathbf{k})$ and $\delta\psi(\mathbf{k})$ are, in turn, driven by the beat terms (in (3) and (4)) that contain the mean field $\langle A \rangle$. The calculational approach here treats fluid and magnetic fluctuations on an equal footing, and seeks to determine Γ_A by probing an evolved state of MHD turbulence, rather than a kinematically prescribed state of velocity fluctuations alone. The calculation follows those of Pouquet *et al.* (1976) and Pouquet (1978), and yields the result

$$\Gamma_A = -\sum_{\mathbf{k}'} \left[\tau_c^{\psi}(\mathbf{k}')\langle v^2 \rangle_{\mathbf{k}'} - \tau_c^{A}(\mathbf{k}')\langle B^2 \rangle_{\mathbf{k}'}\right] \frac{\partial \langle A \rangle}{\partial x} - \sum_{\mathbf{k}'} \left[\tau_c^{A}(\mathbf{k}')\langle A^2 \rangle_{\mathbf{k}'}\right] \frac{\partial}{\partial x}\langle J \rangle. \tag{8}$$

Here, consistent with the restriction to a weak mean field, isotropic turbulence is assumed. The quantities $\tau_c^{\psi}(\mathbf{k})$ and $\tau_c^{A}(\mathbf{k})$ are the self-correlation times (lifetimes), at $\mathbf{k}$, of the fluid and field perturbations, respectively. These are not at all necessarily equivalent to the coherence time of $v_x(-\mathbf{k}')$ with $A(\mathbf{k}')$, which determines Γ_A. For a weak mean field, both $\tau_c^{\psi}(\mathbf{k})$ and $\tau_c^{A}(\mathbf{k})$ are determined by nonlinear interaction processes, so that $1/\tau_c^{\psi,A}(\mathbf{k}') \geq k'\langle B \rangle$, i.e., fluctuation correlation times are *short* in comparison to the Alfvén time of the mean field. In this case, the decorrelation process is controlled by the Alfvén time of the *r.m.s.* field (i.e., $[\mathbf{k}\langle B^2\rangle^{1/2}]^{-1}$) and the fluid eddy turnover time, as discussed by Pouquet *et al*. (1976) and Pouquet (1978). Consistent with the assumption of unity magnetic Prandtl number, $\tau_c^{\psi}(\mathbf{k}) = \tau_c^{A}(\mathbf{k}) = \tau_c(\mathbf{k})$, hereafter.

The three terms on the right-hand side of (8) correspond respectively (Diamond *et al*., 1984) to

(a) a positive turbulent resistivity (i.e., Γ_A proportional to flux gradient) due to fluid advection of flux;
(b) a negative turbulent resistivity symptomatic of the tendency of magnetic flux to accumulate on large scales;
(c) a positive turbulent hyper-resistive diffusion, which gives Γ_A proportional to *current* gradient (Strauss, 1986). Such diffusion of current has been proposed as the mechanism whereby a magnetofluid undergoes Taylor relaxation (Taylor, 1986; Bhattacharjee and Hameiri, 1986; Bhattacharjee and Yuan, 1995).

Note that terms (b) and (c) both arise from $B_x(\mathbf{k})\delta\psi(\mathbf{k}')$, and show the trend in 2D MHD turbulence to pump large-scale H_A while damping small-scale H_A. For smooth, slowly varying mean potential profiles, the hyper-resistive term is negligible in comparison with the turbulent resistivity, (i.e., $\langle k'^2\rangle > (1/\langle A\rangle)(\partial^2\langle A\rangle/\partial x^2)$), so that the mean magnetic potential flux reduces to

$$\Gamma_A = -\eta_T\,\frac{\partial\langle A\rangle}{\partial x}, \tag{9}$$

where

$$\eta_T = \sum_{\mathbf{k}'} \tau_c(\mathbf{k}')\left(\langle v^2\rangle_{\mathbf{k}'} - \langle B^2\rangle_{\mathbf{k}'}\right). \tag{10}$$

As stated above, the critical element in determining Γ_A is to calculate $\langle B^2\rangle_{\mathbf{k}'}$ in terms of $\langle v^2\rangle_{\mathbf{k}'}$, Γ_A itself, etc. For this, mean-square magnetic potential balance is crucial! To see this, note that the flux equation may be written as

$$\frac{\partial A}{\partial t} + \mathbf{v}\cdot\nabla A = -v_x\frac{\partial\langle A\rangle}{\partial x} + \eta\nabla^2 A, \tag{11}$$

so multiplying by A and summing over modes gives

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\langle A^2\rangle + \langle\nabla\cdot(\mathbf{v}A^2)\rangle\right] = -\Gamma_A\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle, \tag{12}$$

assuming incompressibility of the flow. An equivalent, **k**-space version of (12) is

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\langle A^2\rangle_{\mathbf{k}} + T\,(\mathbf{k})\right] = -\Gamma_A(\mathbf{k})\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\mathbf{k}}, \tag{13}$$

where $T\,(\mathbf{k})$ is the triple correlation

$$T\,(\mathbf{k}) = \langle \nabla\cdot(\mathbf{v}A^2)\rangle_{\mathbf{k}}, \tag{14}$$

which controls the nonlinear transfer of mean-square potential, and $\Gamma_A(\mathbf{k}) = \langle v_x A\rangle_{\mathbf{k}}$ is the **k**-component of the flux. Equations (12) and (13) thus allow the determination of $\langle B^2\rangle$ and $\langle B^2\rangle_{\mathbf{k}}$ in terms of Γ_A, $\Gamma_A(\mathbf{k})$, $T\,(\mathbf{k})$ and $\partial\langle A^2\rangle_{\mathbf{k}}/\partial t$.

At the simplest, crudest level (the so-called τ-approximation), a single τ_c is assumed to characterize the response or correlation time in (10). In that case, we have

$$\Gamma_A = -\left[\sum_{\mathbf{k}} \tau_c(\langle v^2\rangle_{\mathbf{k}} - \langle B^2\rangle_{\mathbf{k}})\right]\frac{\partial\langle A\rangle}{\partial x}. \tag{15}$$

For this, admittedly over-simplified case, (12) then allows the determination of $\langle B^2\rangle$ in terms of Γ_A, the triplet and $\partial_t\langle A^2\rangle$. With the additional restrictions of stationary turbulence and periodic boundary conditions (so that $\partial\langle A^2\rangle/\partial t = 0$ and $\langle\nabla\cdot(\mathbf{v}AA)\rangle = 0$), it follows that

$$\langle B^2\rangle = -\frac{\Gamma_A}{\eta}\frac{\partial\langle A\rangle}{\partial x}, \tag{16}$$

so that magnetic fluctuation energy is directly proportional to magnetic potential flux, via H_A balance. This corresponds to a balance between local dissipation and spatial flux in the mean-square potential budget (Gruzinov and Diamond, 1995, 1996). Inserting this into (10) then yields the following expression for the turbulent diffusivity:

$$\eta_T = \frac{\sum_{\mathbf{k}}\tau_c\langle v^2\rangle_{\mathbf{k}}}{1+\tau_c v_{A0}^2/\eta} = \frac{\eta^k}{1+R_m v_{A0}^2/\langle v^2\rangle}, \tag{17}$$

where η^k refers to the kinematic turbulent resistivity $\tau_c\langle v^2\rangle$, v_{A0} is the Alfvén speed of the mean $\langle B\rangle$, and $R_m = \langle v^2\rangle\tau_c/\eta$. It is instructive to note that (17) can be rewritten as

$$\eta_T = \frac{\eta\eta^k}{\eta+\tau_c v_{A0}^2}. \tag{18}$$

Thus, as indicated by mean-square potential balance, Γ_A *ultimately scales directly with the collisional resistivity*, a not unexpected result for Alfvénized turbulence with dynamically interesting magnetic fluctuation intensities. This result supports the intuition discussed earlier. It is also interesting to note that for $R_m v_{A0}^2/\langle v^2\rangle > 1$ and $\langle v^2\rangle \sim \langle B^2\rangle$, $\eta_T \cong \eta\langle B^2\rangle/\langle B\rangle^2$, consistent with the Zeldovich theorem prediction.

Equation (17) gives the well-known result for the quenched flux diffusivity. There, the kinematic diffusivity η_T^k is modified by the quenching or suppression factor $[1+R_m v_{A0}^2/\langle v^2\rangle]^{-1}$, the salient dependencies of which are on R_m and $\langle B\rangle^2$. Equation (17) predicts a strong quenching of η_T with increasing $R_m\langle B\rangle^2$. Despite the crude approximations made in the derivation, numerical calculations indicate remarkably good agreement between the measured cross-field flux diffusivity (as determined by following marker particles tied to a flux element) and the predictions of (17). In particular, the scalings with both R_m and $\langle B\rangle^2$ have been verified, up to R_m values of a few hundred (Cattaneo, 1994).

Of course, the derivation of (17), as well as the conclusion of a quenched magnetic diffusivity, have provoked many questions, together with a vigorous debate in the community, though primarily in the context of directly analogous issues in the 3D alpha-quenching problem. The criticisms leveled at the treatment of α in the 3D problem (e.g., by Blackman and Field, 2000, 2002) must however also carry over to the treatment of β in the 2D case, and so we address this issue here. Criticism has focused primarily upon what is perceived as an inadequate treatment of the triplet term $\langle\nabla\cdot(\mathbf{v}AA)\rangle$ in (12). Note that $\langle\nabla\cdot(\mathbf{v}AA)\rangle$ makes no contribution to global $\mathcal{H}_A$ balance in a periodic system. However, while $\langle\nabla\cdot(\mathbf{v}AA)\rangle = 0$ in this case, $\langle\mathbf{v}\cdot(\nabla AA)\rangle_{\mathbf{k}}$ does not. This contribution to the $\langle A^2\rangle$ dynamics corresponds to

(i) the divergence of the flux of mean-square potential, $\nabla\cdot\boldsymbol{\Gamma}_{A^2}$, (here $\boldsymbol{\Gamma}_{\mathbf{A^2}} = \mathbf{vAA}$), when considered in a region of position space of scale $|\mathbf{k}|^{-1}$;

(ii) the spectral transport of $\langle A^2\rangle_{\mathbf{k}}$, when considered in $\mathbf{k}$-space.

In either case, a new timescale enters the mean-square magnetic potential budget which can, in principle, break the balance between $\Gamma_A\langle B\rangle$ and resistive dissipation. Physically, this timescale has been associated with

(i) the net outflow of mean-square potential at the boundaries, in the case of a non-periodic configuration. In this regard, it has been conjectured that should the loss rate of $\langle A^2\rangle$ exceed that of $\langle A\rangle$, the quench of η_T would be weaker.

(ii) the *local* effective transport rate (on scales $\sim |\mathbf{k}|^{-1}$) of mean-square potential or, alternatively, the local spectral transport rate of $\langle A^2\rangle_{\mathbf{k}}$. Note that in this case, boundary conditions are irrelevant. Thus, *local* $\langle A^2\rangle$ spectral transport effects should manifest themselves in numerical calculations with periodic boundaries, such as those by Cattaneo (1994).

To address these questions, one must calculate the triplet correlations. In this regard, it is instructive to consider them from the point of view of transport in position space (i.e., $\langle \nabla \cdot (\mathbf{v}AA)\rangle$), together with the equivalent spectral transfer in $\mathbf{k}$-space. The goal here is to assess the degree to which triplet correlations enter the relationship between resistive dissipation and magnetic flux transport, which is central to the notion of quenching.

Recall, on retaining the volume-averaged advective flux, that the equation for the mean-square potential fluctuation is

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle AA\rangle + \langle \mathbf{v}\cdot\nabla AA\rangle\right) = -\langle v_x A\rangle \frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle. \tag{19}$$

Observe that since $-\langle v_x A\rangle \partial\langle A\rangle/\partial x = \eta_T\,\langle B\rangle^2$, the right-hand side of (19) simply reduces to the Zeldovich theorem, $\langle B^2\rangle/\langle B\rangle^2 = \eta_T/\eta$, in the absence of contributions from the triplet moment. For stationary turbulence, then, the proportionality between mean flux transport and resistive dissipation is broken by the triplet $\langle \mathbf{v}\cdot\nabla AA\rangle$, which may be rewritten as $\langle \mathbf{v}\cdot\nabla AA\rangle = \nabla\cdot\langle \mathbf{v}AA\rangle = \int \boldsymbol{\Gamma}_{A^2}\cdot \mathrm{d}\mathbf{n}$, using Gauss's law. Here $\boldsymbol{\Gamma}_{A^2} = \mathbf{v}AA$ is the flux of mean-square potential and the integration $\int \mathrm{d}\mathbf{n}$ is normal to a contour enclosing the region of averaging denoted by the bracket. This scale must, of course, be smaller than the mean field scale ℓ_o for consistency of the averaging procedure. Mean-square potential evolution is thus given by

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle AA\rangle + \int \mathrm{d}\mathbf{n}\cdot\boldsymbol{\Gamma}_{A^2}\right) = -\langle v_x A\rangle \frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle, \tag{20}$$

so that the balance of mean flux transport and local dissipation is indeed broken by the *net* in/out flux of mean-square potential to the averaging region. Alternatively, the triplet correlation renders the mean-square potential balance *non-local*. Of course, $\int \mathrm{d}\mathbf{n}\cdot\boldsymbol{\Gamma}_{A^2}$ is determined by the values of the turbulent velocity and potential perturbation on the boundary of the averaging region. The non-local term in the $\mathcal{H}_A$ budget is by no means "small" in any naive sense, either—indeed a straightforward estimate of the ratio of the second term (the A^2 flux) in (20) to the third term gives $(B/\langle B\rangle)(k_0\ell)^{-1}$, where k_0 is a typical perturbation wave vector and ℓ is the scale of the averaging region. As $B/\langle B\rangle \sim \sqrt{R_m} \gg 1$ and $(k_o\ell)^{-1} \leq 1$, this ratio can certainly be large, so the triplet term is by no means *a priori* negligible. However, two caveats are important. First, a *net* influx or outflux is required, these being more suggestive of an externally driven process, rather than one that is spontaneous (i.e., in 3D, of helicity injection rather than a dynamo). Second, the quantity $\mathbf{\Gamma}_{A^2}$ *may not* be calculated kinematically, for exactly the same reasons that the kinematic theory of Γ_A fails so miserably! This latter point is discussed at length, below.

Noting that a *net inflow or outflow* of mean-squared potential is required to break the local balance between resistive dissipation and mean potential transport (i.e., turbulent resistivity), critics (most prominently Blackman and Field, 2000) of the notion of quenching have advanced the suggestion that a net in/out flux $\mathbf{\Gamma}_{A^2}$ of mean-square potential at the system boundary may weaken the quench. Implicit in this suggestion is the idea that $\mathbf{\Gamma}_{A^2}$ will exceed Γ_A, or alternatively, that the in/out flow rate of mean-square potential exceeds that of the mean potential. We shall see below that when $\mathbf{\Gamma}_{A^2}$ and Γ_A are both calculated self-consistently, this is *not* the case. While a definitive numerical test of this hypothesis has yet to be performed, the results of recent numerical calculations that relax the periodic boundary conditions used in earlier studies by prescribing A or $\partial A/\partial y$ on the upper and lower boundaries indicate no significant departure from the predicted effective resistivity quench (Wilkinson and Hughes, 2005). We hasten to add, however, that while these calculations *do* suggest that the dynamics of turbulent transport are insensitive to boundary conditions, they do not actually examine the effects of external magnetic potential injection.

It is also instructive to examine the triplet correlations in **k**-space, as well as in configuration space. Indeed, it is here that the tremendous departure of $\mathbf{\Gamma}_{A^2}$ from kinematic estimates is most apparent. In **k**-space, $\mathcal{H}_A$ evolution is described by

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle AA\rangle_{\mathbf{k}} + T_{\mathbf{k}}\right) = -\langle v_x A\rangle_{\mathbf{k}}\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\mathbf{k}}, \tag{21}$$

where the triplet $T_{\mathbf{k}}$ is just

$$T_{\mathbf{k}} = \langle \mathbf{v}\cdot\nabla AA\rangle_{\mathbf{k}}. \tag{22}$$

In **k**-space, spectral transfer, rather than spatial transport and mixing, breaks the balance between resistive dissipation and turbulent transport. Thus, the key issue is the calculation of $T_{\mathbf{k}}$. This is easily accomplished by standard closure methods as discussed by Pouquet (1978).

Thus, applying EDQNM or DIA-type closures, $T_{\mathbf{k}}$ is straightforwardly approximated as

$$T_{\mathbf{k}} = \sum_{\mathbf{k}'} \left(\mathbf{k}\cdot\mathbf{k}'\times\hat{\mathbf{z}}\right)^2 \theta_{\substack{\mathbf{k},\mathbf{k}'\\ \mathbf{k}+\mathbf{k}'}} \left\{\langle\psi^2\rangle_{\mathbf{k}'} - \left[\frac{|\mathbf{k}'|^2-|\mathbf{k}|^2}{|\mathbf{k}+\mathbf{k}'|^2}\right]\langle A^2\rangle_{\mathbf{k}'}\right\}\langle A^2\rangle_{\mathbf{k}}$$
$$-\sum_{\substack{\mathbf{p},\mathbf{q}\\ \mathbf{p}+\mathbf{q}=\mathbf{k}}} (\mathbf{p}\cdot\mathbf{q}\times\hat{\mathbf{z}})^2\theta_{\mathbf{k},\mathbf{p},\mathbf{q}}\langle\psi^2\rangle_{\mathbf{p}}\langle A^2\rangle_{\mathbf{q}}, \tag{23}$$

where $\theta_{\mathbf{k},\mathbf{p},\mathbf{q}}$ is the triad coherence time $\theta_{\mathbf{k},\mathbf{p},\mathbf{q}} = (1/\tau_{c\mathbf{k}} + 1/\tau_{c\mathbf{q}} + 1/\tau_{c\mathbf{p}})^{-1}$. In (23), the first and third terms represent advection of potential by the turbulent velocity, the first giving a turbulent resistivity, the third incoherent noise. Note that these two contributions conserve $\langle A^2\rangle$ against each other when summed over $\mathbf{k}$. The second term in (23) corresponds to inverse transfer of mean-square potential via flux coalescence. Note that it is negative on large scales ($k^2 < k'^2$), yielding the negative turbulent resistivity, and positive on small scales ($k^2 > k'^2$), giving the positive hyper-resistivity. Observe that the second term is manifestly antisymmetric in $\mathbf{k}$ and $\mathbf{k}'$, and so conserves $\langle A^2\rangle$ individually, when summed over $\mathbf{k}$.

It is immediately clear that, just as in the case of $\langle A\rangle$, $\langle A^2\rangle$ evolution is determined by the competition between advective straining and mixing of iso-A contours, together with the tendency of these flux structures to coalesce to progressively larger scales. This is hardly a surprise, since A and all its moments are frozen into the flow, up to resistive dissipation. Note also that a proper treatment of mean-square potential conservation (i.e., $\sum_{\mathbf{k}} \mathrm{T}_{\mathbf{k}} = 0$) requires that nonlinear noise due to incoherent mode coupling also be accounted for.

Equation (21) can be re-written in the form

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle AA\rangle_{\mathbf{k}} + \widehat{\eta}_{T\,\mathbf{k}}\langle AA\rangle_{\mathbf{k}} - N_{\mathbf{k}}\right) = -\langle v_x A\rangle_{\mathbf{k}}\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\mathbf{k}}, \tag{24}$$

having written

$$\mathrm{T}_{\mathbf{k}} = \widehat{\eta}_{T\,\mathbf{k}}\langle AA\rangle_{\mathbf{k}} - \mathrm{N}_{\mathbf{k}} \tag{25}$$

with

$$\widehat{\eta}_{T\,\mathbf{k}} = \sum_{\mathbf{k}'}(\mathbf{k}\cdot\mathbf{k}'\times\hat{\mathbf{z}})^2\theta_{\substack{\mathbf{k},\mathbf{k}'\\ \mathbf{k}+\mathbf{k}'}}(\langle\psi^2\rangle_{\mathbf{k}'} - \langle A^2\rangle_{\mathbf{k}'}) \tag{26}$$

and

$$N_{\mathbf{k}} = \sum_{\substack{\mathbf{p},\mathbf{q} \\ \mathbf{p}+\mathbf{q}=\mathbf{k}}} (\mathbf{p}\cdot\mathbf{q}\times\hat{\mathbf{z}})^2 \theta_{\mathbf{k},\mathbf{p},\mathbf{q}} \langle \psi^2 \rangle_{\mathbf{p}} \langle A^2 \rangle_{\mathbf{q}}. \tag{27}$$

Note that $\widehat{\eta}_{T\,\mathbf{k}} \to \partial/\partial x(\eta_T\, \partial/\partial x)$ as $\mathbf{k} \to \mathbf{0}$. It is interesting to compare terms on the left- and right-hand side of (24). Nonlinear transfer terms $\sim \langle \nabla \cdot \boldsymbol{\Gamma}_{A^2} \rangle_{\mathbf{k}}$ are $\mathrm{O}(kA^2|v|)$, while mean flux terms are $\mathrm{O}(|vA||\langle B \rangle|)$. Thus, the ratio $|\mathrm{T}_{\mathbf{k}}|/|\langle \mathrm{vA} \rangle_{\mathbf{k}}||\langle \mathrm{B} \rangle| \sim \mathrm{O}(\mathrm{B}/\langle \mathrm{B} \rangle)$. Here, $B/\langle B \rangle \gg 1$, as we are considering a strongly turbulent, weakly magnetized regime. Thus, to lowest order in $(B/\langle B \rangle)^{-1}$, (24) (at stationarity) must reduce to

$$\mathrm{T}_{\mathbf{k}} = 0, \tag{28}$$

so that nonlinear transfer determines the magnetic potential spectrum. In physical terms, this means that $\langle AA \rangle_{\mathbf{k}}$ adjusts to balance nonlinear noise, which is the main source here. We formally refer to this spectrum as $\langle AA \rangle_{\mathbf{k}}^{(0)}$. Note that $\langle AA \rangle_{\mathbf{k}}^{(0)}$ is actually determined, as is usual for spectral transfer processes, by the balance between $N_{\mathbf{k}}$ (incoherent mode coupling) and $\widehat{\eta}_{T_{\mathbf{k}}} \langle AA \rangle_{\mathbf{k}}$ (turbulent dissipation). This guarantees that the net spectral flow rate is constant in $\mathbf{k}$, so $\mathcal{H}_A$ is conserved. Nonlinear noise is critical here (to respect $\mathcal{H}_A$ conservation) and, in fact, constitutes the dominant source for $\langle AA \rangle_{\mathbf{k}}^{(0)}$ when $B/\langle B \rangle \gg 1$. Note that a corresponding calculation for magnetic helicity by Blackman and Field (2002) neglects nonlinear noise. It is interesting to observe that, as a consequence, the classical "mean field electrodynamics" calculation of $\langle v_x A \rangle$ *cannot* be decoupled from the spectral transfer problem for $\langle AA \rangle_{\mathbf{k}}$. This of course follows from the constraint imposed upon the former by $\mathcal{H}_A$ conservation. To next order in $(B/\langle B \rangle)^{-1}$ then, (24) gives

$$0 = -\langle v_x A \rangle_{\mathbf{k}} \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle_{\mathbf{k}}, \tag{29}$$

the solution of which trivially yields $\langle B^2 \rangle_{\mathbf{k}}$. Ultimately, this yields a quenched turbulent resistivity of the form

$$\eta_T = \sum_{\mathbf{k}} \frac{\tau_{c\mathbf{k}} \langle v^2 \rangle_{\mathbf{k}}}{1 + \tau_{c\mathbf{k}} v_{A0}^2/\eta}. \tag{30}$$

Note that this is basically equivalent to the result in (17), with, however, the quench factor varying with $\mathbf{k}$.

Several comments are in order here. First, it cannot be overemphasized that a self-consistent calculation of $\langle \nabla \cdot \boldsymbol{\Gamma}_{A^2} \rangle_{\mathbf{k}}$ is crucial to this conclusion. Such a calculation necessarily must include both nonlinear response and nonlinear noise. A kinematic calculation would leave $\hat{\eta}_{T\,\mathbf{k}} > \eta_T$, which is incorrect. Likewise, neglecting noise would violate $\mathcal{H}_A$ conservation. It is also amusing to note that the question of the relation between $\langle v_x A\rangle_{\mathbf{k}}\langle A\rangle'$ and $\eta\langle B^2\rangle_{\mathbf{k}}$ does *not* hinge upon boundary conditions or inflow/outflow, at all. Hence, the available numerical experiments, already published, constitute a successful initial test of the theory of flux diffusivity quenching in 2D, at least for modest values of R_m and for smooth $\langle A\rangle$ profiles.

It is instructive to return to configuration space now, in order to compare the rates of transport of $\langle A\rangle$ and $\langle AA\rangle$. The analysis given above may be summarized by writing the equations of evolution for $\langle A\rangle$, i.e.,

$$\frac{\partial}{\partial t}\langle A\rangle = \frac{\partial}{\partial x}\left(\eta_T\,\frac{\partial\langle A\rangle}{\partial x}\right), \tag{31}$$

where

$$\eta_T = \sum_{\mathbf{k}'} \tau_{c\mathbf{k}'}(\langle v^2\rangle_{\mathbf{k}'} - \langle B^2\rangle_{\mathbf{k}'}); \tag{32}$$

and for $\langle AA\rangle_{\mathbf{k}}$, i.e.,

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle AA\rangle_{\mathbf{k}} + \widehat{\eta}_{T\,\mathbf{k}}\langle AA\rangle_{\mathbf{k}}\right) = \frac{1}{2}N_{\mathbf{k}} - \langle v_x A\rangle_{\mathbf{k}}\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\mathbf{k}}, \tag{33}$$

where

$$\widehat{\eta}_{T\,\mathbf{k}} = \sum_{\mathbf{k}'}(\mathbf{k}\cdot\mathbf{k}'\times\hat{\mathbf{z}})^2\theta_{\substack{\mathbf{k},\mathbf{k}'\\ \mathbf{k}+\mathbf{k}'}}(\langle\psi^2\rangle_{\mathbf{k}'} - \langle A^2\rangle_{\mathbf{k}'}). \tag{34}$$

Not surprisingly, $\widehat{\eta}_{T\,\mathbf{k}} \to \eta_T\,\nabla^2$ as $\mathbf{k}\to\mathbf{0}$. This is, of course, a straightforward consequence of the fact that the same physics governs the dynamics of $\langle A\rangle$ and $\langle A^2\rangle$ since A is conserved along fluid trajectories, up to resistive dissipation. Hence, the total diffusive loss rates for $\langle A\rangle$ and $\langle A^2\rangle$ are simply $1/\tau_A = \eta_T\,/L_A^2$ and $1/\tau_{A^2} = \eta_T\,/L_{A^2}^2$, where L_A and L_{A^2} are the gradient scale lengths for $\langle A\rangle$ and $\langle AA\rangle$, respectively. Here L_{A^2} is set either by the profile of forcing or injection, or by the profile of $\langle A\rangle$. For the latter, $\tau_A = \tau_{A^2}$ so that preferential loss of $\langle AA\rangle$ is impossible. For the former, inflow of flux at the boundary, say by plasmoid injection, could however decouple L_{A^2} from L_A. In this case, however, the magnetic dynamics are not spontaneous but, rather, strongly driven by external means.

2.4. Turbulent diffusion of flux and field in 3D

In this section, we discuss flux and field diffusion in three dimensions. In 3D, $\mathbf{A}$ is *not* conserved along fluid element trajectories, so the flux diffusion problem becomes significantly more difficult. With this in mind, we divide the discussion of 3D diffusion into two sub-sections; one on turbulent diffusion in 3D reduced MHD (RMHD) (Strauss, 1976), the other on weakly magnetized, full MHD. This progression facilitates understanding, as 3D RMHD is quite similar in structure to 2D MHD, allowing us to draw on the experience and insight gained in the study of that problem.

Flux diffusion in 3D reduced MHD

The reduced MHD equations in 3D are:

$$\frac{\partial A}{\partial t} + (\nabla\psi \times \hat{\mathbf{z}}) \cdot \nabla A = B_0 \frac{\partial \psi}{\partial z} + \eta \nabla^2 A, \tag{35}$$

$$\frac{\partial}{\partial t}\nabla^2\psi + (\nabla\psi \times \hat{\mathbf{z}}) \cdot \nabla\nabla^2\psi = \nu\nabla^2\nabla^2\psi + (\nabla A \times \hat{\mathbf{z}}) \cdot \nabla\nabla^2 A + B_0 \frac{\partial}{\partial z}\nabla^2 A. \tag{36}$$

These equations describe incompressible MHD in the presence of a strong field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, which is externally prescribed and fixed. The "test field" undergoing turbulent diffusion is $\langle \mathbf{B} \rangle = \langle B(x) \rangle \hat{\mathbf{y}}$, where $\langle B(x) \rangle = -\partial\langle A \rangle/\partial x$. Obviously, $\langle B \rangle \ll B_0$ here.

Of course, the presence of a strong $\mathbf{B}_0$ renders 3D RMHD dynamics quite similar (but *not* identical!) to those in 2D. In particular, note that one can define a mean-square magnetic potential in 3D RMHD, i.e.,

$$H_A = \int \mathrm{d}^2x \int A^2 \mathrm{d}z, \tag{37}$$

and that H_A *is conserved up to resistive dissipation and Alfvénic coupling*, so that the fluctuation H_A balance becomes:

$$\frac{1}{2}\frac{\partial H_A}{\partial t} = -\langle v_x A \rangle \frac{\partial \langle A \rangle}{\partial x} - \eta\langle B^2 \rangle + B_0 \left\langle A \frac{\partial \psi}{\partial z} \right\rangle. \tag{38}$$

In contrast to its 2D counterpart (i.e., (5)), H_A balance is achieved by a competition between cross-field transport and resistive dissipation *together with Alfvénic propagation along* B_0 (i.e., observe that the last term in (38) is explicitly proportional to B_0). It is interesting to note, however, that total H_A conservation is broken only by local dissipation (as in 2D) and by a *linear* effect, which corresponds to wave propagation along B_0. Thus, although H_A is not conserved (even as $\eta \rightarrow 0$), the potential equation nonlinearity (i.e., the nonlinearity in (38)) is *still* annihilated in 3D RMHD, as it is in 2D (i.e., $\langle \mathbf{v} \cdot \boldsymbol{\nabla} AA \rangle \rightarrow 0$, up to boundary flux terms)! Hence, the mean-square potential budget is still a powerful constraint on flux diffusion in 3D.

For simplicity and brevity, the discussion of flux diffusion in 3D is limited to the case of constant τ_c. Proceeding as in the previous section straightforwardly yields

$$\Gamma_A = -\tau_c(\langle v^2 \rangle - \langle B^2 \rangle)\frac{\partial \langle A \rangle}{\partial x}. \tag{39}$$

Here, the current diffusivity has been dropped, as for 2D. To relate $\langle B^2 \rangle$ to $\langle v^2 \rangle$ etc., mean-square potential balance and stationarity give

$$\langle B^2 \rangle = \frac{-\langle v_x A \rangle}{\eta}\frac{\partial \langle A \rangle}{\partial x} + \frac{B_0}{\eta}\left\langle A\frac{\partial \psi}{\partial z}\right\rangle. \tag{40}$$

Thus, the new element in 3D is the appearance of Alfvénic coupling (i.e., the last term on the right-hand side) in the H_A balance. This coupling is non-zero only if there is a net directivity in the radiated Alfvénic spectrum, or, equivalently, an imbalance in the two Elsasser populations, which account for the intensity of wave populations propagating in the $\pm\hat{\mathbf{z}}$ directions.

This contribution may be evaluated as before, i.e.,

$$\left\langle A\frac{\partial \psi}{\partial z}\right\rangle = \left\langle A\frac{\partial}{\partial z}\delta\psi\right\rangle + \left\langle \delta A\frac{\partial \psi}{\partial z}\right\rangle, \tag{41}$$

where $\delta\psi$ and δA are obtained via closure of (35, 36). A short calculation gives

$$\left\langle A\frac{\partial \psi}{\partial z}\right\rangle = \tau_c B_0\big(\varepsilon_v \langle v^2 \rangle - \varepsilon_B \langle B^2 \rangle\big), \tag{42}$$

where

$$\varepsilon_v = \frac{\int k_z^2 \langle \psi^2 \rangle_{\mathbf{k}} \mathrm{d}^3 k}{\int \left(k_x^2 + k_y^2\right) \langle \psi^2 \rangle_{\mathbf{k}} \mathrm{d}^3 k}, \tag{43}$$

and ε_B similarly, with $\langle A^2 \rangle_{\mathbf{k}}$. Note that this approximation to $\langle A\partial\psi/\partial z \rangle$ clearly vanishes for equal Elsasser populations with identical spectral structure. This, of course, simply states that, in such a situation, there is no net imbalance or directivity in the Alfvénically radiated energy, and thus no effect on the H_A budget. Taking $\varepsilon_v = \varepsilon_B$ and proceeding as in the 2D case finally yields

$$\Gamma_A = \frac{-\eta_T^k \, \partial\langle A\rangle/\partial x}{(1+\tau_c/\eta)\left(\varepsilon_B B_0^2 + \langle B\rangle^2\right)}, \tag{44}$$

where

$$\eta_T^k = \tau_c \langle v^2\rangle. \tag{45}$$

In 3D, τ_c is also a function of B_0^2, i.e., $\tau_c = \tau_{NL}/(k_z^2 v_A^2 \tau_{NL}^2 + 1)$, where τ_{NL} is the amplitude-dependent correlation time.

The message of (44, 45) is that in 3D RMHD, the strong guide field $\mathbf{B_0}$ contributes to the quenching of η_T. The presence of the factor ε_B implies that this effect is sensitive to the parallel-perpendicular anisotropy of the turbulence, which is eminently reasonable. Thus, the degree of quenching in 3D RMHD is stronger than in 2D, as $B_0 \gg \langle B\rangle$. Finally, note however that the upshot of the quench is still that η_T scales with η, indicative of the effects of the freezing of magnetic potential into the fluid.

Given the attention paid to turbulence energy flux through the system boundary, it is worthwhile to comment here that the Alfvénic radiation contribution to the H_A budget ($\langle A\partial\psi/\partial z\rangle$) could be significantly different if there were a net imbalance in the two Elsasser populations. For example, this might occur in the solar corona, where Alfvén waves propagate away from the Sun, along "open" field-lines. In this case, a local balance between such Alfvénic leakage and cross-field transport could be established in the H_A budget. Such a balance would, of course, greatly change the scalings of η_T from those given here.

An interesting application of mean field electrodynamics within RMHD is to the problem of fast, turbulent reconnection in 3D (Lazarian and Vishniac, 1999; Kim and Diamond, 2001). As, in essence by definition, reconnection rates are measured globally (i.e., over some macroscopic region), they are necessarily constrained by conservation laws, such as that of H_A conservation. It is not surprising, then, that one upshot of the quenching of η_T (i.e., (18)) is that the associated magnetic reconnection velocity $V \leq (\langle v^2\rangle/\langle v_A\rangle^2)^{1/2} v_{s-p}$, where v_{s-p} is the familiar Sweet-Parker velocity $v_{s-p} = \langle v_A\rangle/\sqrt{R_m}$, where $R_m = \langle v_A\rangle L/\eta$. Note that this result states that the reconnection rate is enhanced beyond the prediction of collisional theory, but still exhibits the Sweet-Parker type scaling with resistivity.

Magnetic field diffusion (β-effect) in full, 3D MHD

Moving now to consider the case of weakly magnetized, incompressible, 3D MHD, magnetic potential is no longer conserved, even approximately. Detailed calculations (Gruzinov and Diamond, 1994; Kim, 1999) predict that

$$\eta_T \cong \eta_T^k , \tag{46}$$

or, equivalently, that the kinematic turbulent resistivity is unchanged and unquenched, to leading order. The obvious question then naturally arises as to why α is quenched (see Section 3) but η_T (or, equivalently, β) is not. Here, we note that β being a scalar, and not a pseudo-scalar like α, plays no role in magnetic helicity balance. As magnetic helicity balance, which forces a balance between α and resistive dissipation of magnetic helicity ($\sim \eta\langle \mathbf{B} \cdot \mathbf{J} \rangle$), together with stationarity, is the origin of α-quenching, it is thus not at all surprising that η_T is not quenched in 3D, for weak fields. Of course the weak field result stated here must necessarily pass to the strong field RMHD case discussed earlier, as a strong guiding field is added. The analytical representation of β that smoothly connects these two limiting cases has yet to be derived, and remains an open question in the theory.

Computational studies have not yet really confronted the physics of magnetic flux diffusion in 3D. While two rather minimal studies exist (Thelen and Cattaneo, 2001; Brandenburg, 2001), neither presents systematic scans over R_m or a broad scan over $\langle B \rangle^2$. Although results indicate some tendency toward reduction of β as $\langle B \rangle^2$ increases, it is unclear whether or not the onset of this occurs in the "weak" or "strong" field limit. Further work is clearly needed.

2.5. Discussion and conclusion

In this section, we have reviewed the theory of turbulent transport of magnetic flux and field in 2D and 3D MHD. The 2D flux diffusion problem has been given special attention for its intrinsic interest and relative simplicity, as well as for its many similarities to the problem of the α-effect in 3D. Several issues that are of current interest have been addressed in detail. These include: boundary in-flow and out-flow effects on the mean-square potential budget, the role of nonlinear spectral transfer in the mean-square potential budget, and the dynamics of magnetic flux in 3D reduced MHD. Several topics for further study have been identified, including, but not limited to

(i) the derivation of an expression for diffusion in 3D that unifies the weak and strong field regimes;
(ii) a numerical study of transport in 2D that allows a net flux of turbulence through the system boundary;
(iii) both a theoretical and numerical study of flux diffusion in 3D with balanced and unbalanced Elsasser populations, for various along-field boundary conditions;
(iv) a study of η_T quenching for $P_m \gg 1$ and a consideration of non-stationary states.

Constraints on space force us to omit several extensions and related topics. These include, for example, the turbulent transport of a passive scalar in 2D MHD (Diamond and Gruzinov, 1997), the self-consistent ambipolar diffusion problem in 2D (Kim, 1997; Leprovost and Kim, 2003) and the study of diffusion of magnetic fields in 2D electron MHD (Das and Diamond, 2000; Dastgeer *et al.*, 2000).

3. The generation of magnetic fields

In the previous section we discussed at length the turbulent diffusion of a magnetic field, concentrating primarily on the case of a two-dimensional, coplanar field and flow. The defining feature of this two-dimensional system is that there is no possibility of field *generation*; any large-scale field of zero mean is guaranteed to decay completely, the interest being in the nature and timescale of this decay. Decay in two dimensions is a consequence of the conservation of mean-square magnetic potential. In three dimensions this is no longer the case and, notwithstanding the fascinating similarities between the 2D and 3D cases, represents a major physical difference between them.

Historically, astrophysical interest has been in the behavior of large-scale magnetic fields, as manifested, for example, by the eleven-year solar cycle. Therefore, as explained for the 2D problem, it is natural to seek evolution equations for the large-scale magnetic field involving transport coefficients dependent on properties of the small-scale velocity field and small-scale magnetic field. The simplest approach is to assume a scale separation between the (small) scale of the velocity field, l, and the large scale, L, of the magnetic field. (Although within this framework it is straightforward also to include the effects of a large-scale velocity field, we shall here, for simplicity, assume that the velocity field is only small-scale.) Thus we write

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \tag{47}$$

where $\langle \mathbf{b} \rangle = \mathbf{0}$, angle brackets denoting an average over some intermediate length scale a satisfying $l \ll a \ll L$. By assumption, $\langle \mathbf{V} \rangle = \mathbf{0}$.

3.1. The linear regime

In order to highlight the crucial differences between the two- and three-dimensional problems it is instructive first to consider the kinematic problem, in which the velocity field is prescribed independently of the magnetic field. Averaging the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\mathbf{v} \times \mathbf{B} \right) + \eta \nabla^2 \mathbf{B}, \tag{48}$$

gives

$$\frac{\partial \mathbf{B}_0}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} + \eta \nabla^2 \mathbf{B}_0, \tag{49}$$

where $\boldsymbol{\mathcal{E}} = \langle \mathbf{v} \times \mathbf{b} \rangle$ is the mean electromotive force. Obviously, to make progress with (49) it is necessary to express $\boldsymbol{\mathcal{E}}$ in terms of the mean field $\mathbf{B}_0$ (and its derivatives). The standard approach, developed by Steenbeck, Krause and Rädler in the 1960s (see Krause and Rädler, 1980, for full references), comes from considering the equation for the fluctuating component of the magnetic field, $\mathbf{b}$, obtained by subtracting (49) from the induction equation (48). This gives

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \mathbf{b} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}_0) + \boldsymbol{\nabla} \times \mathbf{G}, \tag{50}$$

where $\mathbf{G} = \mathbf{v} \times \mathbf{b} - \langle \mathbf{v} \times \mathbf{b} \rangle$. The right-hand side of (50) may be interpreted as expressing the two dynamical ingredients contributing to the evolution of the small-scale field $\mathbf{b}$. One is through the mean field $\mathbf{B}_0$ acting as a source for the small-scale field via the $\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}_0)$ term. The other reflects the evolution of the small-scale field *even in the absence of a mean field*. It is instructive to consider these contributions carefully since their understanding lies at the very heart of 3D mean field electrodynamics.

The traditional interpretation of (50) (see, for example, Krause and Rädler, 1980; Moffatt, 1978) has been to assume that the small-scale field $\mathbf{b}$ is driven entirely by the source term $\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}_0)$—in other words, that in the absence of the mean field $\mathbf{B}_0$ the small-scale field would simply decay. From this assumption it becomes possible to construct the extremely elegant theory of mean field electrodynamics. If $\mathbf{b}$ is linearly related to $\mathbf{B}_0$, then, at least for a prescribed flow, $\boldsymbol{\mathcal{E}} = \langle \mathbf{v} \times \mathbf{b} \rangle$ must also be linearly related to $\mathbf{B}_0$. Since the mean field varies on a large length scale, it is natural to posit an expansion for $\boldsymbol{\mathcal{E}}$ of the form

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \cdots \tag{51}$$

As we shall see, for consistency with (49) it is important that the first two terms (and only the first two terms) of this expansion are retained. Since $\boldsymbol{\mathcal{E}}$ is a polar vector whereas $\mathbf{B}$ is an axial vector, it follows therefore that α_{ij} and β_{ijk} are *pseudo*-tensors. The physical interpretation of α_{ij} and β_{ijk} can be most clearly seen for the simple case of isotropic turbulence, for which α_{ij} and β_{ijk}, being dependent only on the properties of the flow, must be isotropic tensors; i.e., $\alpha_{ij} = \alpha \delta_{ij}$ and $\beta_{ijk} = \beta \epsilon_{ijk}$, where α is a pseudo-scalar and β is a true scalar. Substitution from (51) into (49) then gives

$$\frac{\partial \mathbf{B}_0}{\partial t} = \boldsymbol{\nabla} \times \alpha \mathbf{B}_0 + (\eta + \beta) \nabla^2 \mathbf{B}_0, \tag{52}$$

where, for expository purposes, we have made the further simplifying assumption that β is a constant. Clearly β is an additional, turbulent, contribution to the diffusion, essentially as discussed at length in Section 2. The term involving α (the famous "α-effect" of mean field electrodynamics) is of significance only for three-dimensional fields and flows, and represents the possibility of magnetic field generation—i.e., dynamo action. Over the last three decades, a substantial literature has developed through applying the ideas of mean field electrodynamics, as captured by (52) and more complicated versions thereof, to explaining observed astrophysical fields. Although this is a topic of considerable interest, it is beyond the scope of this review, in which we concentrate solely on the fundamental aspects of the transport coefficients.

Simply through parity considerations it is possible to deduce an immensely powerful result that gets to the very heart of the α-effect. Consider the idealized case of isotropic turbulence that is *reflectionally symmetric*; in other words, the fluid motions, on average, possess no *handedness*. For such turbulence, α, which is assumed to depend only on the statistical properties of the flow (and which, by assumption, possesses no handedness), must remain invariant under a change from a description in terms of a right-handed set of axes to one in terms of a left-handed set. Conversely, α, being a pseudo-scalar, must change sign under such a parity transformation. Consequently, we are forced to the conclusion, purely through parity arguments, that α can be non-zero only for turbulence that lacks reflectional symmetry. Although handedness is imbued into practically all astrophysical systems via rotation, with the Coriolis force providing a natural breaking of reflectional symmetry, the significance of its influence will depend on the temporal and spatial scales of the fluid motions. For example, on the Sun, the fluid motions responsible for the large-scale field (i.e., that of the solar cycle) are rotationally influenced, whereas the influence of rotation on the smaller-scale granular and supergranular convection cells observed at the surface, which have a temporal scale that is very short compared to a solar rotation period, is negligible (see Cattaneo and Hughes, 2001).

Determination of the coefficients α and β

Even in the kinematic regime, in which the back-reaction of the Lorentz force is neglected, calculation of α_{ij} and β_{ijk} is not straightforward. Determination of $\mathbf{b}$ from (50) is made difficult owing to the term $\boldsymbol{\nabla} \times \mathbf{G}$, and it is therefore natural to look for circumstances under which this troublesome term can be neglected—an assumption sometimes referred to as the *quasi-linear approximation* or as *first order smoothing*. Let us define τ_c and l_c as the correlation time and correlation length of the turbulence with r.m.s. velocity v. A simple order of magnitude comparison of the terms in (50) shows that for conventional turbulent flows (for which $\tau_c \sim l_c/v$), $\boldsymbol{\nabla} \times \mathbf{G}$ (and $\partial \mathbf{b}/\partial t$ also) can be neglected only provided that $Rm \ll 1$. An alternative scenario in which $\boldsymbol{\nabla} \times \mathbf{G}$ (but not necessarily $\partial \mathbf{b}/\partial t$) can be neglected is the case of small Strouhal number, i.e., $S = v\tau_c/l_c \ll 1$. Under this latter premise two further distinctions can be made, depending on whether $S \ll Rm$ or $S \gg Rm$. For the first of these, putting aside for the moment any mathematical qualms we may have over dropping the diffusive term, α and β can be determined as

$$\alpha = -\frac{1}{3}\tau_c\langle \mathbf{v} \cdot \boldsymbol{\omega} \rangle, \qquad \beta = \frac{1}{3}\tau_c\langle \mathbf{v}^2 \rangle. \tag{53}$$

As foreshadowed by the discussion above, α is dependent on the handedness of the flow, being directly proportional, and of opposite sign, to the helicity; β, on the other hand, depends not on any parity considerations but only on the kinetic energy of the flow.

It should be noted though that from an astrophysical standpoint, neither of the conditions $Rm \ll 1$ or $S \ll 1$ is ever satisfied; typically Rm is immense and, in what may be deemed as "conventional turbulence", $S \sim O(1)$. Consequently, any results obtained under first order smoothing should always be treated with some caution as to their validity when $Rm \gg 1$. Although an analytic theory for turbulence at high Rm remains elusive, it is however possible to make some headway for the case of perfectly conducting fluids (i.e., when η is zero and Rm is formally infinite), for which the magnetic field is frozen into the fluid. The field at any time t can be related to the field at some initial time ($t = 0$, say) through the Cauchy solution

$$B_i(\mathbf{x}, t) = B_j(\mathbf{a}, 0)\partial x_i/\partial a_j. \tag{54}$$

Formal substitution into the expression $\boldsymbol{\mathcal{E}} = \langle \mathbf{v} \times \mathbf{b} \rangle$ then leads to the following expressions (Moffatt, 1978) for α and β (again assuming isotropy):

$$\alpha(t) = -\frac{1}{3}\int_0^t \langle \mathbf{v}^L(\mathbf{a}, t) \cdot \boldsymbol{\nabla}_\mathbf{a} \times \mathbf{v}^L(\mathbf{a}, \tau)\rangle \mathrm{d}\tau, \tag{55}$$

$$\begin{aligned}\beta(t) = {} & \frac{1}{3}\int_0^t \langle \mathbf{v}^L(\mathbf{a}, t) \cdot \mathbf{v}^L(\mathbf{a}, \tau)\rangle \mathrm{d}\tau + \int_0^t \alpha(t)\alpha(\tau)\mathrm{d}\tau \\ & + \frac{1}{6}\int_0^t \int_0^t \langle \mathbf{v}^L(t) \cdot \mathbf{v}^L(\tau_2)\boldsymbol{\nabla}_\mathbf{a} \cdot \mathbf{v}^L(\tau_1) \\ & - (\mathbf{v}^L(t) \cdot \boldsymbol{\nabla}_\mathbf{a}\mathbf{v}^L(\tau_1)) \cdot \mathbf{v}^L(\tau_2)\rangle \mathrm{d}\tau_1 \mathrm{d}\tau_2,\end{aligned} \tag{56}$$

where $\mathbf{v}^L(\mathbf{a}, t)$ is the Lagrangian representation of the velocity at time t of the fluid element located initially at $\mathbf{x} = \mathbf{a}$. As expected, the expression for α reflects the handedness of the flow (expressed now in a Lagrangian sense). Of more surprise is the expression for β; the first term on the right-hand side of (56) is simply the effective turbulent diffusivity of a scalar field, whereas the second and third terms arise exclusively as a consequence of the vector character of $\mathbf{B}$. It is of interest to note that the expression for β contains products of α at different times, suggesting that the handedness of the flow may, at high Rm, be of significance in determining the diffusion of the magnetic field. A word of caution though is in order regarding expressions (55) and (56). First, there is no guarantee of convergence of the integrals contained in these expressions; furthermore, it is not clear if there is a unique interpretation of these results, owing to the fact that for a perfectly conducting fluid the initial state of the magnetic field is never forgotten. That said, they provide a potentially useful insight into the astrophysically relevant, and theoretically most challenging regime of $Rm \gg 1$.

3.2. *Small-scale dynamo action*

The exposition above has been based on the premise that the perturbed field $\mathbf{b}$ owes its existence solely to the large-scale field $\mathbf{B}_0$, and that, consequently, $\mathcal{E}$ is a homogeneous linear functional of $\mathbf{B}_0$ and its derivatives. Recent studies have however revealed that turbulent flows (exhibiting exponential separation of particle trajectories) typically act as *small-scale* dynamos—with the magnetic field having scales comparable to or smaller than that of the driving flow—at sufficiently high values of Rm (see, for example, the monograph by Childress and Gilbert, 1995). Indeed, such flows (depending on their stretching and folding properties) can act as dynamos even in the limit of $Rm \to \infty$ (so-called *fast* dynamos). The case of astrophysical relevance is thus most likely to be that for which the growth of a large-scale magnetic field $\mathbf{B}_0$ is considered in the presence of a small-scale field that can exist independently of $\mathbf{B}_0$. In this case, only part of the small-scale field $\mathbf{b}$ in (50) owes its existence to $\mathbf{B}_0$, and we expect $\mathcal{E}$ to be an *inhomogeneous* linear functional of $\mathbf{B}_0$. The possibility of small-scale dynamo action was clearly recognized by the pioneers of mean field dynamo theory (see, for example, Krause and Rädler, 1980) though it is only fairly recently that the pervasiveness of small-scale dynamo action at high Rm has been fully appreciated.

3.3. *The nonlinear regime*

Although obtaining an understanding of even the kinematic evolution of a 3D magnetic field is not straightforward—and remains far from complete—it is nonetheless important to address the problem of the nature of magnetic field transport in the nonlinear regime, i.e., when the back-reaction of the field on the flow cannot be neglected. The most important issue is to determine, for high values of Rm, the strength of the large-scale field at which α and β differ significantly from their kinematic values. A variety of approaches to this problem has been undertaken, based on the conservation laws of the ideal (diffusionless) system, on closure arguments, on numerical simulation of the governing equations or on some combination of these. The present state of play is that these differing approaches have not yet provided an agreed solution, making this one of the most controversial topics in astrophysical MHD. In this section we shall concentrate principally on the nature of the α-effect, the current understanding of turbulent diffusion in 3D being on even less solid ground. The physics of magnetic diffusion in 2D was extensively discussed in Section 2.

In order to calculate α it is sufficient to imagine an experiment in which homogeneous turbulence is permeated by a *uniform* magnetic field $\mathbf{B}_0$. From simple physical considerations one expects that as the strength of the imposed field is increased, and the Lorentz forces become significant, the strength of the α-effect will be reduced—so-called *α-suppression*. Given the symmetry of the system under a change in sign of $\mathbf{B}_0$, we may therefore expect, at high values of Rm, a dependence of the form

$$\alpha = \mathcal{F}\left(Rm^{\gamma} B_0^2/B_E^2\right), \tag{57}$$

where $\mathcal{F}$ is a decreasing function of B_0^2, and where B_E, denoting the equipartition strength of the large-scale field, is a useful reference measure of the field strength. The simplest such formula, which is often adopted, is

$$\alpha = \frac{\alpha_0}{1 + Rm^{\gamma} B_0^2/B_E^2}, \tag{58}$$

where α_0 represents the kinematic value. The controversial nature of the subject resides in the value of the exponent γ. If γ is extremely small then large-scale fields close to equipartition strength can be generated before the α-effect loses its efficiency; conversely if γ is O(1) then the generation term shuts down when the large-scale field is still extremely weak. Whereas certain theories of MHD turbulence may suggest the former alternative, numerical simulations—backed up by theoretical interpretation—point most decidedly to the latter.

The dependence of α on Rm and B_0 can be determined unambiguously—at least for a range of values of Rm and B_0—by numerical solution of the nonlinear MHD equations for a turbulent flow permeated by a uniform magnetic field. Cattaneo and Hughes (1996) and Cattaneo *et al*. (2002) have considered this problem for a flow driven by helical forcing. It is worth reiterating that α is a statistical quantity and hence has a meaningful value only when averaged correctly; this point is illustrated in Fig. 3, which shows that although α clearly has a well-defined long-term mean, averaging over too short an interval could lead to quite erroneous results. The results of the simulations, illustrated in Fig. 4, are compatible only with an O(1) value of the exponent γ, i.e., they show a dramatic α-suppression.

In a series of complementary calculations, Cattaneo *et al*. (2002) considered the evolution of a magnetic field of zero mean (i.e., no imposed field) in an extended spatial domain. Reassuringly, from the point of view of mean field theory, the nature of the α-effect driving the growth

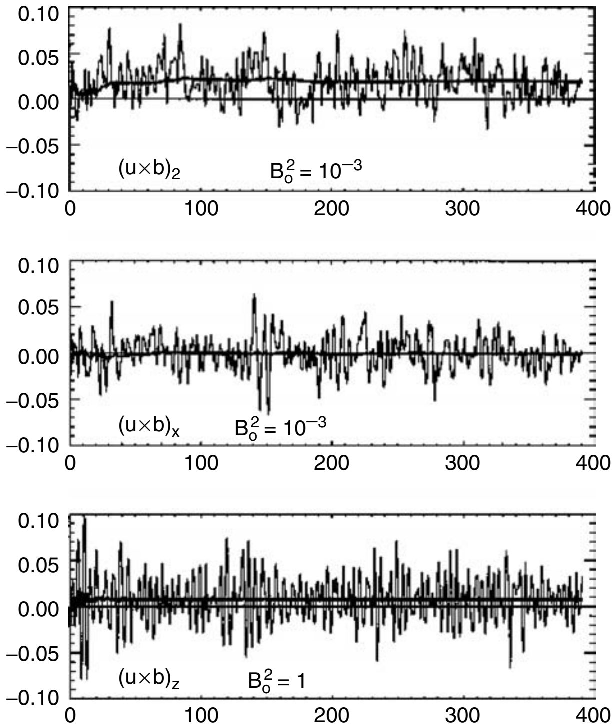

Figure 3. Time histories and time averages (thick lines) of components of the e.m.f. for a helically forced turbulent flow with an imposed uniform field (from Cattaneo and Hughes, 1996). In units of the equipartition strength, $B_0^2 = 10^{-3}$ in the uppermost panels, and $B_0^2 = 1$ in the lower panel. The e.m.f. (and hence α) has sizeable temporal variations, but a well-defined time average.

of the largest-scale field possible is entirely consistent with that derived from the calculations of α from an imposed uniform field; i.e., the growth of the large-scale field is halted at a very low value ($\mathrm{O}(B_E/\sqrt{Rm})$).

It is of course important to address the physical cause of the drastic α-suppression found in these numerical simulations. As discussed earlier, for two-dimensional turbulence the suppression of β can be traced to the fact that the strong small-scale field imbues the fluid particles with a "memory" (see Cattaneo, 1994); this inhibits their tendency to disperse via random walking, and consequently reduces the diffusion of the magnetic field. For three-dimensional magnetic fields we envisage a similar physical picture. For high values of Rm, it is indubitable that strong small-scale fields ($\mathrm{O}(Rm^{1/2}B_0)$) are generated, even while

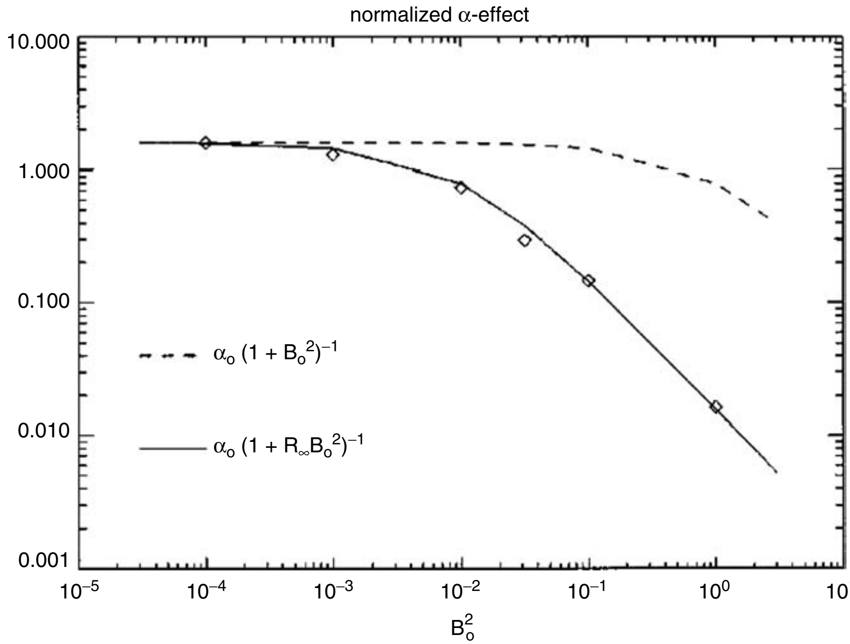

Figure 4. Normalized α-effect as a function of imposed field energy B_0^2 for a helically forced flow with $Rm = 100$ (from Cattaneo and Hughes, 1996). The dashed and solid lines are fits to the data for the two forms of α-quenching indicated.

the large-scale field is weak. Consequently, we may expect a marked reduction in Lagrangian transport properties and hence in α. It is though fair to say that a detailed numerical calculation of α via its Lagrangian properties has yet to be performed.

Whereas high-resolution numerical simulations point most definitely toward a dramatic (or even "catastrophic" in certain eyes) suppression of α, it is of course important to consider alternative approaches to the problem. One such approach, already discussed in Section 2, is through the use of turbulence closure models, following their success at reproducing many of the features of hydrodynamic (non-magnetic) turbulence. The most widely used scheme is the EDQNM model of Pouquet *et al*. (1976), the magnetic version of the scheme proposed by Orszag (1970). The key result of their analysis is the derivation of an expression for α in the following form

$$\alpha = -\frac{1}{3}\tau_c(\langle \mathbf{v}\cdot\boldsymbol{\omega}\rangle - \langle \mathbf{j}\cdot\mathbf{b}\rangle), \tag{59}$$

which subsequently has been widely used. It is however not only worth bearing in mind that this is a result borne of a number of approximations and assumptions—such as the assumption that the correlation times for the velocity and magnetic fields are the same—but it is also worth discussing how the result fits in with the classical α-effect picture described above. As discussed by Proctor (2003), the fact that the induction equation remains linear in the magnetic field—even though in the dynamic regime the flow is of course affected by the field—simply leads to the usual quasi-linear result (53). Any nonlinearity is simply manifested in a change to the kinetic helicity distribution. So what is the origin of the second term in (59)? If, instead of the classical picture of $\mathbf{b}$ being dependent on $\mathbf{B}_0$, we consider the introduction of a large-scale field $\mathbf{B}_0$ into a *pre-existing* state of MHD turbulence with a small-scale velocity $\mathbf{v}$ and a small-scale field $\mathbf{b}$ — leading to further perturbations $\mathbf{v}'$ and $\mathbf{b}'$ — then, under the quasi-linear approximation,

$$\mathcal{E} = \langle \mathbf{v} \times \mathbf{b}' \rangle + \langle \mathbf{v}' \times \mathbf{b} \rangle, \tag{60}$$

which, in combination with the momentum equation, leads to the result (59) (Pouquet *et al.*, 1976; Kleeorin and Ruzmaikin, 1982; Gruzinov and Diamond, 1994, 1996; Kleeorin and Rogachevskii, 1999; Proctor, 2003). It is though vitally important to be clear about the exact meanings of $\mathbf{v}$ and $\mathbf{b}$ in this formula. To obtain a further insight into the α-effect it is instructive to consider the ideal topological invariant (Gruzinov and Diamond, 1994, 1996). Whereas the physics of the diffusion of a magnetic field in two dimensions is underpinned by the conservation (in the absence of diffusion) of the mean-square potential, in three dimensions the conserved quantity is not $\langle \mathbf{A}^2 \rangle$, but the magnetic helicity $\langle \mathbf{A} \cdot \mathbf{B} \rangle$. The equations for $\mathbf{a}$ and $\mathbf{b}$, the perturbations of the vector potential and the magnetic field, are

$$\frac{\partial \mathbf{a}}{\partial t} = (\mathbf{v} \times \mathbf{B}_0) + (\mathbf{v} \times \mathbf{b}) - \boldsymbol{\nabla} \phi - \eta \boldsymbol{\nabla} \times \mathbf{b}, \tag{61}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}_0) + \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b}. \tag{62}$$

If it is assumed that the small scales are stationary, such that $\partial \langle \mathbf{a} \cdot \mathbf{b} \rangle / \partial t = 0$, then (61) and (62), subject to reasonable boundary conditions, imply that

$$\mathbf{B}_0 \cdot \langle \mathbf{v} \times \mathbf{b} \rangle = \mathbf{B}_0 \cdot \mathcal{E} = -\eta \langle \mathbf{b} \cdot \boldsymbol{\nabla} \times \mathbf{b} \rangle = -\eta \langle \mathbf{j} \cdot \mathbf{b} \rangle, \tag{63}$$

and consequently we have the *exact* result, dependent only on stationarity, that

$$\alpha = -\frac{\eta\langle \mathbf{j}\cdot\mathbf{b}\rangle}{B_0^2}, \tag{64}$$

where $\mathbf{b}$ is the entire small-scale magnetic field. If we equate the two expressions for $\langle \mathbf{j}\cdot\mathbf{b}\rangle$ from (59) and (64) then we obtain the strong suppression result (58), though it is worth reiterating that in (64) $\mathbf{b}$ refers to the total small-scale field, whereas in (59) it refers to a pre-existing small-scale field. It is interesting to note that (64) is the analogue for α, in 3D, of the Zeldovich theorem in 2D. In particular, it establishes a *direct* proportionality between α and the collisional resistivity, just as the Zeldovich theorem demonstrates the direct proportionality of η_T to η. Also, note that taking $\gamma = 1$ and $R_m B_0^2/B_E^2 \gg 1$ in (58), along with the assumption of Alfvénized turbulence (so that $\mathbf{v} \sim \mathbf{b}$, $\boldsymbol{\omega} \sim \mathbf{j}$), recovers (64).

3.4. *The role of boundary conditions*

In a series of papers, Blackman, Field and co-workers have raised a number of different—though sometimes self-contradictory—objections to the idea that the nonlinear dependence of the α-effect on the strength of the large-scale field should involve the magnetic Reynolds number in a critical manner. Field *et al.* (1999) claimed that the strong suppression result found by Cattaneo and Hughes (1996) was simply incorrect, though gave no explanation as to why they thought this might be the case. However it should be noted that a number of the assumptions of Field *et al.* (1999) are highly questionable and that, accordingly, the validity of their analysis is in doubt. A year later Blackman and Field (2000) changed the nature of their objection, arguing—in a sharp contradiction of Field *et al.* (1999)—that the strong (Rm-dependent) suppression was, after all, correct, but was however not applicable to astrophysical situations. Instead they claim that the suppression of α found in calculations such as those of Cattaneo and Hughes (1996) "is not a dynamic suppression" and that the suppression occurs entirely through the choice of periodic boundary conditions. The first of these assertions is patently false; the suppression results entirely from the action of the Lorentz force and is thus as dynamical as it can be! The second point, at least couched in the form that the choice of boundary conditions may be important, is however at least worthy of exploration. In Section 2 we considered this very issue for the simpler 2D case and showed how the issue of strong suppression of turbulent diffusion could not easily be dismissed simply through changing the boundary conditions. Here we look at what is involved for the 3D case.

From Ohm's law we can readily derive (for uniform $\mathbf{B}_0$) the *exact* result

$$\alpha B_0^2 = \boldsymbol{\mathcal{E}} \cdot \mathbf{B}_0 = -\frac{1}{\sigma}\langle \mathbf{j} \cdot \mathbf{b} \rangle + \langle \mathbf{e} \cdot \mathbf{b} \rangle, \tag{65}$$

where σ is the electrical conductivity. As discussed above, under certain assumptions it is possible, simply from the $\langle \mathbf{j} \cdot \mathbf{b} \rangle$ term, to derive the strong suppression result. So the interesting question is whether this is the dominant term or if it can be eclipsed by $\langle \mathbf{e} \cdot \mathbf{b} \rangle$. From (61) and (62) it follows that the small-scale magnetic helicity satisfies

$$\frac{\partial \langle \mathbf{a} \cdot \mathbf{b} \rangle}{\partial t} = -2\langle \mathbf{e} \cdot \mathbf{b} \rangle + \langle \boldsymbol{\nabla} \cdot (\mathbf{b}\phi) \rangle - \langle \boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{e}) \rangle, \tag{66}$$

where ϕ is the electrostatic potential. The divergence terms can of course be readily transformed into surface integrals. For periodic boundary conditions the surface terms will vanish; it then follows that, for stationary turbulence, $\langle \mathbf{e} \cdot \mathbf{b} \rangle$ must also vanish. Thus, for stationary turbulence and periodic boundary conditions, α depends only on $\langle \mathbf{j} \cdot \mathbf{b} \rangle$. The thrust of Blackman and Field's argument appears to be that this is a rather special case and that, for other boundary conditions, $\langle \mathbf{e} \cdot \mathbf{b} \rangle$ will dominate and that the whole picture of α-suppression will be changed. Although this is an interesting suggestion it remains, at the moment, nothing more. For the 2D diffusion problem, discussed in Section 2, we saw that periodic boundary conditions are not particularly special and that the surface integral in question vanishes for most "reasonable" boundary conditions. Similarly, for the 3D problem the surface terms may still vanish for reasonable non-periodic conditions. Even more importantly, the mere fact that the surface integrals may not vanish does not, of itself, invalidate the strong-suppression results. Indeed it is by no means obvious, *a priori*, what the magnitude or even the sign of the surface terms will be. Furthermore, as discussed earlier, there is a good, *local* physical argument to explain the strong suppression of α. Any argument that claims the strong suppression result is an artefact of the choice of boundary conditions must also explain away the physical explanation of strong suppression. That said, the precise role of the boundary conditions remains an interesting issue that needs to be properly explored.

4. Momentum and flux transport in 2D MHD

4.1. Overview

In the previous sections, we have reviewed the status of mean field theory for flux diffusion in 2D MHD and for the α-effect in 3D MHD. Here we discuss the transport of momentum and magnetic potential in 2D MHD, incorporating a mean background shear flow. In view of the widespread occurrence of large-scale magnetic fields and shear flows in astrophysical objects—such as in the solar tachocline, accretion discs and galaxies—an outstanding problem in astrophysical MHD is to determine how these two structures influence one another; i.e., how magnetic fields alter the evolution of a mean shear flow via momentum transport, and how a shear flow affects the diffusion of magnetic fields via magnetic flux transport. The introduction of a mean shear flow into an MHD system presents us with rich and complex dynamics. Thus, for the sake of simplicity, the discussion here is limited to 2D MHD, with a mean shear flow parallel to the mean magnetic field; furthermore, a magnetic Prandtl number of unity is assumed. The related problem of momentum transport in 3D RMHD is treated by Kim *et al*. (2001). Given the constraints on space, we do not address the many works on the problem of momentum or angular momentum transport *per se*, as captured by the anisotropic kinetic alpha (AKA) effect (Frisch *et al*., 1987), or the Λ-effect (Rüdiger, 1989; Kichatinov and Rüdiger, 1993).

4.2. Mean field theory

We consider forced 2D MHD turbulence in the (x, y) plane, in which energy is injected on small scales by an external forcing F. The evolution of the magnetic field $\mathbf{B}$ and the fluid velocity $\mathbf{v}$, in terms of the vector potential A ($\mathbf{B} = \boldsymbol{\nabla} \times A\hat{\mathbf{z}}$) and the vorticity ω ($\boldsymbol{\nabla} \times \mathbf{v} = \omega\hat{\mathbf{z}}$), is described by the equations:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\omega = -(\mathbf{B}\cdot\boldsymbol{\nabla})\nabla^2 A + \nu\nabla^2\omega + F, \tag{67}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)A = \eta\nabla^2 A, \tag{68}$$

where F is the external small-scale forcing.

We assume that both the mean shear flow $\mathbf{V}$ and mean magnetic field $\mathbf{B}$ are in the y-direction, with $\mathbf{V} = V(x)\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{y}}$ (or $A = A(x)$).

Adopting a two-scale analysis, we decompose fields into mean and fluctuating components as $\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}' = \mathbf{V} + \mathbf{v}'$, $\omega = \langle \omega \rangle + \omega' = \partial V/\partial x + \omega'$, $\mathbf{b} = \langle \mathbf{b} \rangle + \mathbf{b}' = \mathbf{B} + \mathbf{b}'$ and $a = \langle a \rangle + a' = A + a'$, where angular brackets denote an average over the statistics of the forcing. Note that in this section, the mean fields and fluctuations are denoted by capital letters and primes, respectively. Employing a quasi-linear closure then gives the equations for fluctuations as

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y} - \nu\nabla^2\right)\omega' = -\left(B\frac{\partial}{\partial x}\right)\nabla^2 a' + F, \tag{69}$$

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y} - \eta\nabla^2\right)a' = -\left(v_x'\frac{\partial}{\partial x}\right)A. \tag{70}$$

The fluctuating fields influence the evolution of mean fields via the fluxes of magnetic potential $\langle v_x' a' \rangle$ and momentum $\langle v_x' v_y' - b_x' b_y' \rangle$, which appear in the mean field equations as follows:

$$\left(\frac{\partial}{\partial t} - v\frac{\partial^2}{\partial x^2}\right)V = -\frac{\partial}{\partial y}\langle \Pi \rangle - \frac{\partial}{\partial x}\langle v_x' v_y' - b_x' b_y' \rangle, \tag{71}$$

$$\left(\frac{\partial}{\partial t} - v\frac{\partial^2}{\partial x^2}\right)A = -\frac{\partial}{\partial x}\langle u_x' a' \rangle, \tag{72}$$

where Π is the total pressure. The flux of magnetic potential $\langle v_x' a' \rangle$ in (72), which was discussed in Section 2, represents the effective dissipation rate of a mean magnetic field due to small-scale fluctuations, and can be expressed as $\langle v_x' a' \rangle = -\eta_T\, \partial A/\partial x$, on invoking a turbulent diffusivity η_T. As noted earlier, it consists of two competing processes, which transport magnetic potential to small and large scales via fluid advection and the Lorentz force, respectively. With energy being injected into the fluid, as assumed here, the former process wins, resulting in the overall dissipation of a mean magnetic field through a positive η_T. The dissipation of a mean magnetic field is in fact a natural consequence of the absence of dynamo action in 2D MHD. However, if there is a magnetic forcing in a system, for instance via winds from stars carrying magnetic fields, negative η_T (i.e., an inverse transfer of magnetic potential) is possible (Kim and Dubrulle, 2002).

The momentum flux or total stress $\langle v_x' v_y' - b_x' b_y' \rangle$ in (71) represents the average y-component of flux of the x-component of momentum and captures the overall effect of small-scale fluctuations on the evolution of the mean shear flow V. Acting as an effective force on the mean shear flow, it modifies the mean profile of background shear; it can be put in the form of a turbulent viscous force by introducing a transport coefficient, the so-called turbulent viscosity ν_T, as $\langle v'_x v'_y - b'_x b'_y \rangle = -\nu_T \, \partial V/\partial x$. It can, in general, take either sign, being negative in the case of an inverse cascade and positive for a direct cascade. Note that momentum transport involves the Maxwell stress $\langle b'_x b'_y \rangle$ as well as the usual fluid Reynolds stress $\langle v'_x v'_y \rangle$.

In order for the two turbulent transport coefficients (η_T and ν_T) not to vanish in 2D MHD, non-ideal effects (i.e., irreversibility) in the system are absolutely crucial. One obvious example of such an effect is dissipation, which makes the system deviate from the Alfvénic state into which **B** naturally forces it. The flux of magnetic potential, based on this effect, was reviewed earlier. The presence of a shear flow brings in another non-ideal effect through resonance between the flow and fluctuations (i.e., critical layers). As we shall see, transport of magnetic potential and momentum is reduced by a shear flow as well as by magnetic fields, although the presence of a shear flow itself is critical to a non-vanishing flux of momentum. As the net effect of either a shear flow or a magnetic field on momentum and flux transport is difficult to ascertain, we first discuss the effect of a shear flow on transport in general, and how it may be incorporated non-perturbatively, before discussing transport in 2D MHD with shear (e.g., the effect of shear flow on flux diffusion and the effect of magnetic fields on momentum transport). The results are summarized in Table 2.

4.3. *The effect of shear flows on transport*

A shear flow acts to tilt and elongate eddies, resulting in finer scales as time progresses (see Fig. 5); i.e., for a flow $V(x)\hat{\mathbf{y}}$, the wavenumber k_x grows linearly in time as $k_x(t) = k_x(0) - tk_y \partial V/\partial x$ with constant $k_y(t) = k_y(0)$ (Goldreich and Lynden-Bell, 1965). For an incompressible fluid, as assumed here, fluid velocities perpendicular to the shear are

Table 2. Summary of η_T and ν_T for turbulence with background shear

$\nu = \eta$	Strong shear ($\xi \gg 1$)	Weak shear ($\xi \ll 1$)
2D HD	$\nu_T \propto -1/\nu^2$	$\nu_T \propto -1/\Omega^2$
2D MHD ($\gamma \to 0$)		
Kinematic limit	$\eta_T \propto 1/\nu^2$	$\eta_T \propto 1/\Omega^2$
2D MHD ($\gamma \gg 1$)	$\eta_T \propto 1/\nu^2$	$\eta_T \propto (\nu/\Omega)^{2/3}/B^2$
Strong B	$\eta_T \propto 1/B^2$	$\eta_T \propto 1/B^2$

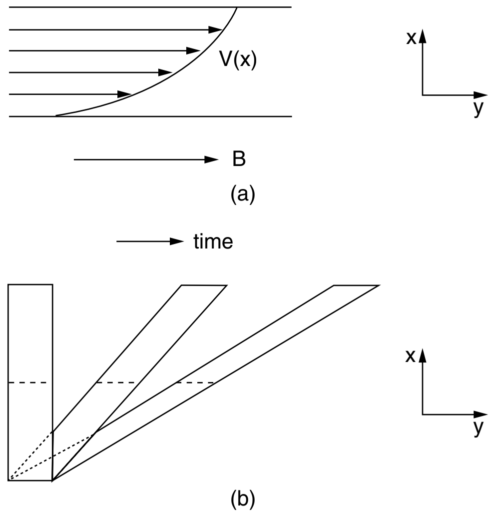

Figure 5. Panel (a) depicts the configuration of a mean shear flow $V(x)\hat{\mathbf{y}}$ and a mean magnetic field $B\hat{\mathbf{y}}$. Panel (b) illustrates the tilting of an eddy by a shear $V(x)\hat{\mathbf{y}}$. The solid and dashed lines represent an eddy in the absence and presence of the field $B\hat{\mathbf{y}}$. The x-extent of the eddy is smaller in the presence of $B\hat{\mathbf{y}}$ since the Lorentz force prevents the eddy motion in the x direction. Tilting elongates an eddy in the y direction, generating small scales in the x direction as time progresses.

smaller than those along the flow, effectively reducing the perpendicular transport of scalar fields. Furthermore, as the perpendicular scale (i.e., that in the x-direction) decreases, the eddies will eventually be torn apart by dissipation, again inhibiting the transport in the x-direction. The reduction of transport by shearing is a common phenomenon, occurring in various physical systems, such as heat transport in geophysical convection (Or and Busse, 1987), and particle and heat transport in magnetically confined plasmas (Biglari *et al.*, 1990; Diamond *et al.*, 1998; Kim and Diamond, 2003).

Owing to the generation of fine scales, a careful, non-perturbative analysis is desirable to capture the effects of shearing. For instance, the effect of shearing on dissipation can be critical to determining the transport, since the overall dissipation, due to this shearing, increases in time. Thus, even if the dissipation may be negligible at some instant, this may not be the case for all subsequent times. The linear increase in wavenumber can be best incorporated non-perturbatively by following a particle trajectory in an extended phase space $(\mathbf{x}, \mathbf{k}, t)$ along which $k_x(t)$ evolves as $k_x(t) = k_x(0) + tk_y \partial V/\partial x$. As shall be shown shortly, this can be achieved via the Gabor transform (a kind of wavelet transform), defined as:

$$\mathrm{GT}[A(\mathbf{x},t)] \equiv \hat{A}(\mathbf{k},\mathbf{x},t) \equiv \int \mathrm{d}^2x' f(|\mathbf{x}-\mathbf{x}'|)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}A(\mathbf{x}',t), \quad (73)$$

where $f(x)$ is a filter function with a characteristic scale λ lying between the characteristic scales of fluctuating and mean fields, thus screening out information outside a domain of size λ. For simplicity, we adopt a Gaussian filter with $f(x) = \exp(-x^2/\lambda^2)$.

In terms of Gabor transforms, (69) and (70) for the fluctuations can be written as

$$\left[\frac{\mathrm{D}}{\mathrm{D}t} + \nu(k^2+p^2)\right]\hat{\omega} = \mathrm{i}Bk(k^2+p^2)\hat{a} + \hat{\mathrm{F}}, \quad (74)$$

$$\left[\frac{\mathrm{D}}{\mathrm{D}t} + \eta(k^2+p^2)\right]\hat{a} = \frac{\mathrm{i}k}{k^2+p^2}\hat{\omega}B = \hat{u}_y B. \quad (75)$$

Here, $\mathbf{k} = (p,k,0)$ and $\hat{\mathbf{v}}(\mathbf{x},\mathbf{k},t) = (\hat{v}_x,\hat{v}_y,0)$; $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + V\,\partial/\partial y - k\partial/\partial x(V\,\partial/\partial p) = \partial/\partial t + V\,\partial/\partial y + k\Omega\partial/\partial p$ is the total time derivative, which includes the linear increase of k_x in time, i.e., $\mathrm{D}y/\mathrm{D}t = V$, $\mathrm{D}x/\mathrm{D}t = 0$, $\mathrm{D}k/\mathrm{D}t = 0$, $\mathrm{D}p/\mathrm{D}t = k\Omega$; without loss of generality, $\Omega = -\partial V/\partial x$ is assumed to be positive. Thus, ray equations along particle trajectories are simply $x = x_0$, $y = y_0 + V(t-t_0)$, $p = p_0 + k\Omega(t-t_0)$, and $k = k_0$, where a subscript "0" denotes the initial value at $t = t_0$. Equations (74) and (75), together with the ray equations, describe the motions of the wave packets of vorticity and magnetic potential in phase space, under the action of large-scale fields. The center of the wave packet is advected at the mean velocity, while its wavenumber is varied according to the local shear, with its amplitude changed by mutual interactions and forcing.

The turbulent transport coefficients follow from the solutions to (74) and (75), which can be found along a particle trajectory for a given forcing F. Therefore, the solutions for ω' and a', and consequently the values of the transport coefficients η_T and ν_T, depend on the properties of the forcing. While it is interesting to study these dependencies, the following discussion focuses on only one special, but non-trivial case, for which the statistics of the forcing are homogeneous in space with an infinitesimally short temporal correlation time (δ-correlated forcing). One interesting property of δ-correlated forcing containing the entire range of frequencies in the spectrum, is that it allows resonance between a shear and fluctuations over a wide range of frequencies. Before discussing the effect of shear and magnetic fields on η_T and ν_T it is illuminating to study how shearing affects transport in general, through the examples of the transport of passive scalar fields and momentum transport in 2D hydrodynamic turbulence (HD).

As an instructive example of the reduction of the transport of passive scalar fields, let us consider the transport of magnetic potential in 2D MHD in the kinematic limit, in which the field is passively advected; i.e., we consider the coupled equations (67) and (68) (or (74) and (75)) but neglect the Lorentz force term. Assuming that the system settles into a stationary state in the long-time limit, we can investigate how the flux of magnetic potential $\langle v_x' a\rangle$ depends on the shear Ω in this stationary state.

Depending on timescales, there are two interesting cases to consider. The first is that of a weak shear, with $\xi = \nu k^2/\Omega \gg 1$, for which the effect of dissipation dominates that of shearing, with the dissipation timescale $\tau_\nu = 1/\nu k^2$ being much smaller than the characteristic shear timescale, $\tau_s = 1/\Omega$; here k refers to the wavenumber of the prescribed forcing. The second is the case of strong shear, with $\xi = \nu k^2/\Omega \ll 1$, in which the effect of shearing dominates that of dissipation. Note that $\xi \ll 1$ can be satisfied even in the long-time limit, despite the generation of fine scales for a' and ω'.

For δ-correlated forcing, turbulent diffusivity in the weak shear case $\xi \gg 1$ is given by

$$\eta_T = \frac{1}{4(2\pi)^2}\int d^2k\,\frac{\hat{\psi}(\mathbf{k})}{\nu^2 k^6}\left[1-\frac{4\Omega^2}{(\nu k^2)^2}\right], \tag{76}$$

while in the strong shear case $\xi \ll 1$, it has the form

$$\eta_T = \frac{1}{32\Omega^2}\int d^2k\,\frac{\hat{\psi}(\mathbf{k})}{k^2}, \tag{77}$$

where $\hat{\psi}(\mathbf{k})$ is the power spectrum in Fourier space of the forcing F. In both limits, the turbulent diffusivity, being positive, is a decreasing function of Ω, confirming the reduction of flux transport by shear. Note that in the strong shear case, the scaling of $\eta_T \propto \Omega^{-2}$ in (77) is a consequence of the δ-correlated forcing, and may thus change for a forcing with different statistical properties.

Momentum transport in 2D HD

Shear also leads to a reduction in the transport of a vector quantity such as momentum. For example, in 2D HD, the turbulent viscosity in the strong shear limit $\xi \gg 1$ is given by

$$\nu_T \sim -\frac{1}{2\Omega^2(2\pi)^2}\int d^2k\,\frac{\hat{\phi}(\mathbf{k})}{k^2}. \tag{78}$$

The negative sign of ν_T is the manifestation of an inverse cascade of energy in 2D HD due to the conservation of enstrophy; the amplitude of ν_T, decreasing with Ω, indicates the suppression of the momentum flux by a strong shear. Observe that when the Ω^{-2} dependence of ν_T is substituted into the mean field equation (71), one recovers the logarithmic equilibrium profile for a mean shear flow (Kim and Dubrulle, 2001). Note that the foregoing calculations assumed a (local) periodic box for fluctuations, while proper wall boundary conditions for a mean flow in (71) are critical to obtaining this equilibrium profile.

4.4. The effect of shear flows on flux diffusion in 2D MHD

As discussed in Section 2, flux diffusion is reduced owing to the back-reaction of the Lorentz force, with an enormous suppression factor for large magnetic Reynolds R_m. The presence of a shear flow introduces an additional non-ideal effect (i.e., irreversibility) via the stochasticity of fluid elements in the presence of resonance between a shear flow and fluctuations—*critical layers*—and the overlap of these layers. Note that this stochasticity may justify our quasi-linear analysis. In the strong shear limit, this resonant absorption leads to flux transport that decreases with increasing shear. Nevertheless, being independent of R_m, this additional effect offers the possibility of weakening the strong dependence of the flux diffusion on R_m.

The exact form of flux diffusion, which depends on magnetic fields and shear, can be obtained from (74) and (75). While it is a formidable task to obtain general solutions to these equations, simple analytical solutions are available in the limit of a strong magnetic field, for which the Alfvén wave timescale of the k mode (associated with a mean magnetic field) is smaller than the timescale for shearing, i.e., $\gamma = |Bk/\Omega| \gg 1$. Once such solutions are found, they can be used to obtain the turbulent diffusivity, yielding the results:

$$\eta_T = \frac{1}{4(2\pi)^2} \int d^2k \, \frac{\hat{\phi}(\mathbf{k})}{k^2 + p^2} \frac{1}{(\nu k^2)^2} \frac{1}{1 + \Lambda}, \tag{79}$$

for the weak shear case with $\xi \gg 1$, and

$$\eta_T = \frac{\nu}{4B^2(2\pi)^2} \int d^2k \, \frac{\hat{\phi}(\mathbf{k})}{k^2 + p^2} \frac{2}{3} \Gamma\left(\frac{1}{3}\right) \left(\frac{3}{2\nu k^2}\right)^{1/3} \Omega^{-2/3}, \tag{80}$$

for the strong shear case $\xi \ll 1$. Here, $\Lambda = (B/\nu k)^2$, $\Gamma(x)$ is the Gamma function, and $\hat{\phi}(\mathbf{k})$ is the power spectrum of the forcing, which is assumed to be homogeneous, but not necessarily isotropic. In both cases, the turbulent diffusivity is positive to leading order, becoming small as the magnetic field B becomes strong. As discussed previously, this is due to the Alfvénization of turbulence by a strong field. It may also be viewed as the inhibition of the eddy motion in the y direction by magnetic tension, effectively reducing the effect of tilting by the shear, as demonstrated by the dashed line in Fig. 5b. In the weak shear limit $\xi \gg 1$, the diffusion of magnetic field is dominated by the strong field, with (79) coinciding with the kinematic result (76) to leading order for $\Lambda \ll 1$.

In the more interesting case of a strong shear, the scaling of $\eta_T \propto \Omega^{-2/3}B^{-2}$ in (80) reveals the interesting, combined effect of shear and magnetic field on the turbulent diffusion of magnetic field. Note that this particular scaling is a consequence of the assumption $\gamma \gg 1$ (which implies the dominance of the effect of magnetic field compared to the shearing), possibly as well as the assumption of δ-correlated forcing. As η_T is primarily suppressed by a magnetic field, the dependence of η_T on Ω is much weaker than that in the kinematic case (see (77)) where $\eta_T \sim \Omega^{-2}$. Nevertheless, in this limit, the amplitude of η_T is smaller than the kinematic value, roughly by a factor of $\xi^{2/3}/\gamma^2$. Note that this factor is proportional to $V^2/B^2R_m^{2/3}$, where V is the characteristic velocity. The R_m dependence of this suppression factor is weak compared to the case without shear, as shown below.

In the absence of a shear flow, the usual quasi-linear analysis of (69) and (70) via Fourier analysis yields the kinematic turbulent diffusivity

$$\eta_k \sim \frac{\tau_f \langle F^2 \rangle}{8\nu^2 k_0^6} \sim \frac{\langle v^2 \rangle}{2\nu k_0^2}. \tag{81}$$

Here, τ_f and k_0 are the correlation time and characteristic scale of the forcing F, respectively, and the relation $\langle v^2 \rangle \sim \tau_f \langle F^2 \rangle / 4\nu k_0^4$ has been used. On the other hand, in the limit of a strong magnetic field, the turbulent magnetic diffusivity is given by

$$\eta_T \sim \frac{\tau_f \langle F^2 \rangle}{4k_0^4 B^2} \sim \frac{\eta \langle v^2 \rangle}{B^2}. \tag{82}$$

Thus, without shear, η_T is reduced by a factor of $2(\eta k_0/B)^2 = (\xi/\gamma)^2 \sim (U/BR_m)^2$ by a magnetic field (U is a typical velocity). Here, note that compared to the suppression factor of v^2/B^2R_m in Kim (1999), the extra factor of $R_m (= R_e)$ comes from the R_e dependence of the kinematic value of turbulent diffusivity (recall, $\eta = \nu$ is assumed), and has nothing to do with a dynamical effect of the Lorentz force. Therefore, $\eta_T \propto R_m^{-3/2}$

with a shear flow and $\eta_T \propto R_m^{-2}$ without it. Therefore, the presence of a shear flow weakens the R_m dependence of η_T through resonance.

We may check that the Zeldovich theorem $\eta_T = \eta\langle b'^2\rangle/B^2$ is valid in the strong shear case (recall $\nu = \eta$) (Zeldovich, 1957)—as indeed it must be. To see this, we note that, to leading order, kinetic $\langle v'^2\rangle$ and magnetic energies $\langle b'^2\rangle$ are given by

$$\langle v'^2\rangle = \langle b'^2\rangle = \frac{1}{6(2\pi)^2\Omega}\int \mathrm{d}^2k\,\frac{\hat{\phi}(\mathbf{k})}{k^2}\Gamma\left(\frac{1}{3}\right)\left(\frac{3\Omega}{2\nu k^2}\right)^{1/3}. \qquad (83)$$

The divergence of $\langle v'^2\rangle$ and $\langle b'^2\rangle$ as $\nu \to 0$ is caused by the accumulation of energy on small scales due to the direct cascade of energy with the (small-scale) forcing. Therefore, $\eta_T = \eta\langle b'^2\rangle/B^2$, with the Zeldovich theorem remaining valid for a strong shear. Note that the weaker dependence of η_T on R_m results from the fact that $\langle b'^2\rangle$ itself depends on η, diverging as $\eta \to 0$.

4.5. Effect of magnetic fields and shear on momentum transport

We now discuss the effect of both magnetic fields and shear flows on momentum transport, starting with the effect of magnetic fields. In 2D HD, enstrophy is an ideal invariant, causing an inverse cascade of energy. Introducing magnetic fields into the 2D HD system breaks this conservation law, since vorticity can be generated by the Lorentz force. Therefore, one of the important effects of magnetic field on momentum transport is to alter the direction of the energy cascade. The other interesting consequence of including magnetic fields is the change in the dependence of the momentum flux with shear. In 2D HD the momentum flux is reduced solely by the shear, while in 2D MHD it is suppressed by both shear and magnetic field. The reduction in momentum transport by a mean magnetic field is due to the Alfvénization of turbulence, and is indicated by the appearance, with opposite signs, of both the Reynolds and Maxwell stresses in the momentum flux. It is reminiscent of the flux of magnetic potential $\Gamma_A \propto \langle v'^2 - b'^2\rangle$, for which perfect Alfvénization leads to $\Gamma_A = 0$. Thus, a significant reduction in momentum transport is possible via a cancelation between Reynolds and Maxwell stresses. In fact, for Alfvén waves, a perfect cancelation between the two stresses is expected. Obviously, the presence of a shear flow (which is necessary for momentum transport in the first place) breaks this perfect Alfvénic state, and leads to a finite momentum transport. As a matter of fact, in the strong shear limit, each of the Reynolds and Maxwell stresses diverges in the ideal limit, but the total stress remains finite owing to the cancelation between the two stresses. The deviation from a pure Alfvénic state can also be achieved by the incorporation of dissipation in the system.

Expressions for the turbulent viscosity follow from the solutions to the coupled equations (74) and (75) with the following results:

(i) In the weak shear case $\xi \gg 1$:

$$\nu_T = \frac{1}{4B^2(2\pi)^2} \int \mathrm{d}^2k \frac{\hat{\phi}(\mathbf{k})}{k^2(k^2+p^2)} \frac{\Lambda(\Lambda-1)}{(1+\Lambda)^2}, \tag{84}$$

where $\Lambda \equiv (B/\nu k)^2$. The direct cascade of energy is already indicated in this weak shear case by a positive ν_T when $\Lambda \gg 1$, in contrast to a negative ν_T in 2D HD. As noted previously, this is a consequence of the Lorentz force, which relaxes the conservation of vorticity constraint in 2D HD, thereby reversing the direction of the energy cascade from inverse to direct. The cancelation between Reynolds and Maxwell stresses, as a result of Alfvénization, is suggested in the amplitude of the turbulent viscosity, which becomes small for a strong magnetic field.

(ii) In the strong shear case $\xi \ll 1$:

$$\nu_T = \frac{1}{4B^2(2\pi)^2} \int \mathrm{d}^2k \frac{\hat{\phi}(\mathbf{k})}{k^2(k^2+p^2)}, \tag{85}$$

which is now always positive, indicating the direct cascade of energy for a strong shear. Furthermore, it is independent of the shear. Note that while $\xi < 1$ for shearing to be of interest (Alfvén frequency less than shear rate), γ can be either greater or less than unity. For $\gamma \gg 1$, the regime of strong magnetization, it is thus not surprising that suppression of ν_T occurs primarily via Alfvénic coupling. When the expression for ν_T is substituted into (71), an equilibrium mean shear flow is found to have either parabolic or linear profiles, depending on the boundary condition for the average total pressure $\langle \Pi \rangle$. Therefore, another important effect of a magnetic field is to change the profile of a mean equilibrium shear flow from logarithmic to parabolic or linear. This is suggestive of the so-called *buffer layer*, which has been studied in the context of turbulent drag reduction in various laboratory experiments (Tsinober, 1989). Note that turbulent viscosity here is the same as that in the weak shear case for $\Lambda \gg 1$, simply because the magnetic field is the main source for the suppression of momentum transport in both cases.

4.6. Concluding remarks

Mean magnetic fields and shear flow are ubiquitous structures in many astrophysical objects, and, as such, the problems of the evolution of these mean magnetic fields (dynamos or diffusion) and of the transport of (angular) momentum are by far two of the most important issues in astrophysical MHD. While it is necessary to consider fully 3D MHD, the primary focus of this section has been to elucidate the role of mean shear flows and magnetic fields on momentum transport and magnetic diffusion, by considering 2D MHD with a mean flow parallel to the magnetic field. The main conclusions are:

(i) magnetic fields have a significant effect on momentum transport—via Alfvénization—leading to the suppression of momentum transport and to laminarization of an equilibrium shear flow;

(ii) a shear flow can weaken (slightly) a problematic strong R_m dependence of magnetic flux diffusion by introducing a route to collisionless irreversibility via resonance overlap.

These results may have significant implications for the solar tachocline where a mean shear flow (provided by the radial differential rotation) is aligned with a strong toroidal magnetic field (see the review by Tobias, 2004, Chapter 7 in this volume). In the solar tachocline, $\gamma = Bk/\Omega \gtrsim 1$ since $B \sim 10^4 - 10^5$G, $\Omega \sim V/L \sim 10^{-6}\text{s}^{-1}$ ($L \sim 10^{10}$cm and $V \sim 10^4$cm/s), and $k > 10^{-10}\text{cm}^{-1}$. Thus, if we take the molecular values for ν and η as $\nu \sim 10^4\text{cm}^2$/s and $\eta \sim 10^2\text{cm}^2$/s, our results obtained in the strong shear limit ($\xi = \nu k^2/\Omega \ll 1$), together with the assumption $\gamma \gg 1$, are applicable in this region, although the assumption of unit magnetic Prandtl number ($\nu = \eta$) is not rigorously justified. Indeed, the solar tachocline is a very interesting site for both dynamo action and angular momentum transport. In order to combat the problems of diffusivity quenching by a strong field, discussed earlier, Parker (1993) proposed the idea of an *interface dynamo*, in which the sites of generation of toroidal field (via a velocity shear) and poloidal field (via the α-effect) are spatially separated, but coupled by turbulent diffusion. Obviously, in the light of the results outlined in this section, it would be of interest to investigate this coupling further, by quantifying the effect of the velocity shear on the diffusion of magnetic field. Furthermore, a possible implication of the result (ii) for 3D MHD is that the incorporation of a shear flow may weaken the notorious α quench. This issue should be investigated by extending the analysis to three dimensions.

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Appendix A: Derivation of the Zeldovich theorem

The Zeldovich theorem is, in essence, a statement of the balance between transport of magnetic flux and resistive dissipation for a stationary, 2D magnetofluid. The Zeldovich theorem may be derived from the evolution equations for the magnetic potential fluctuation A, namely

$$\frac{\partial A}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} A = -v_x \frac{\partial \langle A \rangle}{\partial x} + \eta \nabla^2 A. \tag{A.1}$$

Multiplying by A and summing over space for an incompressible flow yields:

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle A^2\rangle + \langle \boldsymbol{\nabla} \cdot (\mathbf{v}A^2)\rangle\right) = -\langle v_x A\rangle \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle. \tag{A.2}$$

Here, we assume that the spatial variation of the fluctuations is faster than that of the mean potential $\langle A \rangle$, so that $\partial\langle A \rangle/\partial x$ falls outside the brackets in the first term on the RHS of (A.2). For a periodic domain, or one for which $v_n = 0$ on the boundaries, and a stationary state, (A.2) may be re-written as

$$\langle B^2 \rangle = \frac{-\langle v_x A \rangle}{\eta} \frac{\partial \langle A \rangle}{\partial x} = -\frac{\Gamma_A}{\eta} \frac{\partial \langle A \rangle}{\partial x}. \tag{A.3}$$

Finally, writing Γ_A in Fick's law form yields

$$\langle B^2 \rangle = \frac{\eta_T}{\eta}\left(\frac{\partial\langle A\rangle}{\partial x}\right)^2 = \frac{\eta_T}{\eta}\langle B\rangle^2. \tag{A.4}$$

Equation (A.4) is effectively the statement of the Zeldovich theorem.

Equation (A.4) has several interpretations and implications. First, it indicates that the effective turbulent resistivity η_T must scale directly with the collisional resistivity η, in proportion to $\langle B^2\rangle/\langle B\rangle^2$. This, of course, is a straightforward consequence of the freezing-in law, to which the magnetic potential evolution equation is equivalent. Second, (A.4) states that the mean-square magnetic fluctuation $\langle B^2\rangle$ level can be large even if the mean magnetic field $\langle B\rangle$ is weak. Note that $\eta_T/\eta \sim (R_m)_{\text{eff}} \gg 1$, so that $\langle B^2\rangle/\langle B\rangle^2 \gg 1$. Third, (A.4) may be viewed as a statement of Prandtl mixing-length theory for magnetic potential. This follows from the fact that it states an equality between the decay rate of the mean potential ($\sim\eta_T\,(\partial\langle A\rangle/\partial x)^2$ — i.e., the dissipation rate on large scales) and the decay rate of the potential fluctuations ($\sim\eta\langle\nabla A^2\rangle = \eta\langle B^2\rangle$ — i.e., the dissipation rate on small scales). Such a relation constitutes an important constraint on the mean magnetic flux transport, $\mathbf{\Gamma}_A$.

It is useful to mention here that the Zeldovich theorem is a very robust result, which persists in the presence of a mean shear flow, etc. This is, of course, a consequence of the fact that it is basically a straightforward consequence of magnetic flux conservation, or, equivalently, the freezing-in law.

References

Bhattacharjee, A. and Hameiri, E., "Self-consistent dynamo-like activity in turbulent plasmas," *Phys. Rev. Lett.* **57**, 206–209 (1986).

Bhattacharjee, A. and Yuan, Y., "Self-consistent constraints on the dynamo mechanism," *Astrophys. J.* **449**, 739–744 (1995).

Biglari, H., Diamond, P. H. and Terry, P. W., "Influence of sheared poloidal rotation on edge turbulence," *Phys. Fluids B* **2**, 1–4 (1990).

Biskamp, D. and Welter, H., "Dynamics of decaying two-dimensional magnetohydrodynamic turbulence," *Phys. Fluids B* **1**, 1964–1979 (1989).

Blackman, E. G. and Field, G. F., "Constraints on the magnitude of alpha in dynamo theory," *Astrophys. J.* **534**, 984–988 (2000).

Blackman, E. G. and Field, G. F., "New dynamical mean field dynamo theory and closure approach," *Phys. Rev. Lett.* **89**, 265007-1–265007-4 (2002).

Brandenburg, A., "The inverse cascade and nonlinear alpha-effect in simulations of isotropic helical hydromagnetic turbulence," *Astrophys. J.* **550**, 824–840 (2001).

Cattaneo, F., "On the effect of a weak magnetic field on turbulent transport," *Astrophys. J.* **434**, 200–205 (1994).

Cattaneo, F. and Hughes, D. W., "Nonlinear saturation of the turbulent α-effect," *Phys. Rev. E* **54**, R4532–4535 (1996).

Cattaneo, F. and Hughes, D. W., "Solar dynamo theory: a new look at the origin of small-scale magnetic fields," *Astron. Geophys.* **42**, 3.18–3.22 (2001).

Cattaneo, F. and Vainshtein, S. I., "Suppression of turbulent transport by a weak magnetic field," *Astrophys. J.* **376**, L21–L24 (1991).

Cattaneo, F., Hughes, D. W. and Kim, E., "Suppression of chaos in a simplified nonlinear dynamo model," *Phys. Rev. Lett.* **76**, 2057–2060 (1996).

Cattaneo, F., Hughes, D. W. and Thelen, J.-C., "The nonlinear properties of a large-scale dynamo driven by helical forcing," *J. Fluid Mech.* **456**, 219–237 (2002).

Childress, S. and Gilbert, A. D., *Stretch, Twist, Fold: The Fast Dynamo*, Springer-Verlag, Berlin (1995).

Das, A. and Diamond, P. H., "Theory of two-dimensional mean field electron magnetohydrodynamics," *Phys. Plasmas* **7**, 170–177 (2000).

Dastgeer, S., Das, A., Kaw, P. and Diamond, P. H., "Whistlerization and anisotropy in two-dimensional electron magnetohydrodynamics," *Phys. Plasmas* **7**, 571–579 (2000).

Diamond, P. H. and Gruzinov, A. V., "Suppression of cross-field turbulent transport of passive scalar concentration in two dimensional magnetohydrodynamics," *Phys. Rev. Lett.* **78**, 3306–3309 (1997).

Diamond, P. H., Hazeltine, R. D., An, Z. G., Carreras, B. A. and Hicks, H. R., "Theory of anomalous tearing mode growth and the major tokamak disruption," *Phys. Fluids* **27**, 1449–1462 (1984).

Diamond, P. H. *et al.*, "Dynamics of zonal flows and self-regulating drift-wave turbulence," In: *Plasma Phys. and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1998) IAEA-CN-69/TH3/1.

Dubrulle, B., Laval, J.-P. and Sullivan, P., "A new subgrid model for the atmospheric surface layer. II. Prediction of mean flow and velocity and temperature variances," *J. Atmos. Sci.* **59**, 873–887 (2002).

Field, G. F., Blackman, E. G. and Chou, H. S., "Nonlinear alpha-effect in dynamo theory," *Astrophys. J.* **513**, 638–651 (1999).

Frisch, U., She, Z. S. and Sulem, P.-L., "Large-scale flow driven by the anisotropic kinetic alpha effect," *Physica D* **28**, 382–392 (1987).

Fyfe, D. and Montgomery, D., "High-beta turbulence in two-dimensional magnetohydrodynamics," *J. Plasmas Phys.* **16**, 181–191 (1976).

Goldreich, P. and Lynden-Bell, D., "II. Spiral arms as sheared gravitational instabilities," *Mon. Not. R. Astron. Soc.* **130**, 125–158 (1965).

Goldreich, P. and Sridhar, S., "Toward a theory of interstellar turbulence. II. Strong Alfvénic turbulence," *Astrophys. J.* **438**, 763–775 (1995).

Goldreich, P. and Sridhar, S., "Magnetohydrodynamic turbulence revisited," *Astrophys. J.* **485**, 680–688 (1997).

Grappin, R., Pouquet, A. and Léorat, J., "Dependence of MHD turbulence spectra on the velocity field-magnetic field correlation," *Astron. Astrophys.* **126**, 51–58 (1983).
Gruzinov, A. V. and Diamond, P. H., "Self-consistent theory of mean field electrodynamics," *Phys. Rev. Lett.* **72**, 1651–1654 (1994).
Gruzinov, A. V. and Diamond, P. H., "Self-consistent mean field electrodynamics of turbulent dynamos," *Phys. Plasmas* **2**, 1941–1946 (1995).
Gruzinov, A. V. and Diamond, P. H., "Nonlinear mean field electrodynamics of turbulent dynamos," *Phys. Plasmas* **3**, 1853–1857 (1996).
Iroshnikov, T. S., "Turbulence of a conducting fluid in a strong magnetic field," *Sov. Astron.* **7**, 566–571 (1964).
Kichatinov, L. L. and Rüdiger, G., "Λ-effect and differential rotation in stellar convection zones," *Astron. Astrophys.* **276**, 96–102 (1993).
Kim, E., "Turbulent diffusion of large-scale magnetic fields in the presence of ambipolar drift," *Astrophys. J.* **477**, 183–195 (1997).
Kim, E., "Nonlinear dynamo in a simplified statistical model," *Phys. Lett. A* **259**, 232–239 (1999).
Kim, E. and Diamond, P. H., "On turbulent reconnection," *Astrophys. J.* **556**, 1052–1065 (2001).
Kim, E. and Diamond, P. H., "Effect of mean flow shear on cross-phase and transport, reconsidered," *Phys. Rev. Lett.* **91**, 075001-1–075001-4 (2003).
Kim, E. and Dubrulle, B., "Turbulent transport and equilibrium profile in 2D MHD with background shear," *Phys. Plasmas* **8**, 813–824 (2001).
Kim, E. and Dubrulle, B., "Are the energy and magnetic potential cascades direct or inverse in 2D MHD turbulence?," *Physica D* **165**, 213–227 (2002).
Kim, E., Hahm, T. S. and Diamond, P. H., "Eddy viscosity and laminarization of sheared flow in 3D reduced magnetohydrodynamic turbulence," *Phys. Plasmas* **88**, 3576–3582 (2001).
Kleeorin, N. and Rogachevskii, I., "Magnetic helicity tensor for an anisotropic turbulence," *Phys. Rev. E* **59**, 6724–6729 (1999).
Kleeorin, N. and Ruzmaikin, A. A., "Dynamics of the average turbulent helicity in a magnetic field," *Magnetohydrodynamics* **18**, 116–122 (1982).
Kraichnan, R. H., "Inertial-range spectrum of hydromagnetic turbulence," *Phys. Fluids* **8**, 1385–1389 (1965).
Krause, F. and Rädler, K.-H., *Mean Field Electrodynamics and Dynamo Theory*, Pergamon Press (1980).
Lazarian, A. and Vishniac, E., "Reconnection in a weakly stochastic field," *Astrophys. J.* **517**, 700–718 (1999).
Leprovost, N. and Kim, E., "Self-consistent mean field theory in weakly ionized media," *Astrophys. J. Lett.* **598**, L99–102 (2003).
McComb, W. D., *The Physics of Fluid Turbulence*, Oxford University Press, Oxford (1990).
Moffatt, H. K., *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, Cambridge (1978).

Or, C. R. and Busse, F. H., "Convection in a rotating cylindrical annulus. Part 2. Transitions to asymmetric and vacillating flow," *J. Fluid Mech.* **174**, 313–326 (1987).

Orszag, S. A., "Analytical theories of turbulence," *J. Fluid Mech.* **41**, 363–386 (1970).

Parker, E. N., "A solar dynamo surface wave at the interface between convection and uniform rotation," *Astrophys. J.* **408**, 707–719 (1993).

Pouquet, A., "On two-dimensional magnetohydrodynamic turbulence," *J. Fluid Mech.* **88**, 1–16 (1978).

Pouquet, A., Frisch, U. and Léorat, J., "Strong MHD helical turbulence and the nonlinear dynamo effect," *J. Fluid Mech.* **77**, 321–354 (1976).

Proctor, M. R. E., "Dynamo processes: the interaction of turbulence and magnetic fields," in: *Stellar Astrophysical Fluid Dynamics* (Eds. M. J. Thompson and J. Christensen-Dalsgaard), Cambridge University Press, (2003).

Riyopoulos, S., Bondeson, A. and Montgomery, D., "Relaxation toward states of minimum energy in a compact torus," *Phys. Fluids* **25**, 107–115 (1982).

Rüdiger, G., *Differential Rotation and Stellar Convection: Sun and Solar-type Stars*, Gordon and Breach, New York (1989).

Strauss, H. R., "Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks," *Phys. Fluids* **19**, 134–140 (1976).

Strauss, H. R., "Hyper-resistivity produced by tearing mode turbulence," *Phys. Fluids* **29**, 3668–3671 (1986).

Taylor, J. B., "Relaxation and magnetic reconnection in plasmas," *Rev. Mod. Phys.* **58**, 741–763 (1986).

Thelen, J.-C. and Cattaneo, F., "Preliminary measurement of the turbulent magnetic diffusivity in three dimensions," in: *Dynamo and Dynamics, a Mathematical Challenge* (Eds. P. Chossat, D. Armbruster and I. Oprea), pp. 101–108, Kluwer (2001).

Tobias, S. M., "The Solar Tachocline: formation, stability and its role in the solar dynamo," this volume, Chapter 7.

Tsinober, A. B., in: *Viscous Drag Reduction in Boundary Layers, Progress in Astronautics and Aeronautics* (Eds. D. M. Bushnell and J. N. Hefner), Vol. 123, p. 327, American Institute of Aeronautics and Astronautics, Washington D. C. (1989).

Wilkinson, L. and Hughes, D. W., "The role of boundary conditions in turbulent magnetic diffusion," in preparation (2005).

Zeldovich, Ya. B., "The magnetic field in the two dimensional motion of a conducting turbulent liquid," *Sov. Phys. JETP* **4**, 460–462 (1957).